TESTS FOR CONVERGENCE

Suppose you are given and infinite sequence of terms a_0, a_1, a_2, ... The notation $\sum_{n=0}^{\infty} a_n$ tells you to add all these terms, as follows: Form the partial sums

$$\begin{aligned} S_0 &= a_0 \\ S_1 &= a_0 + a_1 \\ &\vdots \\ S_N &= a_0 + a_1 + \cdots + a_N = \sum_{n=0}^{N} a_n \end{aligned}$$

and then take the *limit* of these sums as $N \to \infty$. If this limit exists and is a number, then $\sum_{n=0}^{\infty} a_n$ is called a **convergent infinite series**. If the limit does not exist, or is $\pm\infty$, the series is said to **diverge**.

WARNING: There are *two* sequences here:

(i) The *terms* of the series, a_0, a_1, a_2, ..., and

(ii) the *partial sums* S_0, S_1, S_2, ..., formed by adding more and more of those terms.

With infinite series, it is the limit of the partial sums that concerns us, but to decide if a series converges or diverges, i.e. to decide whether this limit exists, we will often isolate and look at the terms of the series *as a sequence only*, rather than their sum. It is important that you understand this distinction.

OPERATIONS ON SERIES

1. $\displaystyle\sum_{n=0}^{\infty} c\, a_n = c \sum_{n=0}^{\infty} a_n$, for any constant c.

2. If both $\displaystyle\sum_{n=0}^{\infty} a_n$ and $\displaystyle\sum_{n=0}^{\infty} b_n$ converge, then $\displaystyle\sum_{n=0}^{\infty} (a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$.

Note: There are no formulas for $\displaystyle\sum_{n=0}^{\infty} a_n b_n$ or $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{b_n}$.

IMPORTANT SERIES

1. Geometric: $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, where a is any constant and $-1 < r < 1$. If r is outside of this interval, the geometric series diverges. Try to understand the proof of these statements. It will help you understand what partial sums are, how to take their limit, and it will make you remember what their formula is in this particular case.

2. Harmonic and Alternating Harmonic: $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$. The harmonic series diverges (integral test), while the alternating harmonic series converges (alternating series test).

3. p-series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is a constant. This series converges if $p > 1$ and diverges if $p \le 1$. Note that the harmonic series is a p-series with $p = 1$.

You must memorize the above series and when they converge or diverge. Notice that, in a lot of cases, we do not actually know what these series *converge to* when they do converge. This is usually not important. You will mostly only be asked to determine if a given series converges or not, and only if your series looks like the geometric will you be able to say anything more. Also note that some series start with index $n = 0$ and some with $n = 1$ because the term $n = 0$ might not be defined. Where the index starts does not affect the convergence. Throwing away or adding in a finite number of terms does not affect the behavior of an infinite sum.

TESTS FOR CONVERGENCE

1. nth Term Test for Divergence:

$$\text{If } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum a_n \text{ diverges.}$$

Do not turn this around! If $\lim_{n\to\infty} a_n = 0$, you cannot say anything about the convergence of the series. For example, look at the p-series for different p's.

For tests 2–4, $\sum a_n$ and $\sum b_n$ have to be series with positive terms.

2. Integral Test:

Suppose that $a_n = f(n)$, where f is a continuous, positive, and decreasing function. Then

$$\sum_{n=k}^{\infty} f(n) \quad \text{and} \quad \int_k^{\infty} f(x)\,dx$$

either both converge or diverge.

Integral test is very powerful but is useless if you do not know how to integrate $f(x)$. But if you do (for example, if you notice that there is a simple u-substitution hiding in the series), by all means use the Integral Test. Notice that you need to know how to deal with improper integrals (the ones with ∞ in their bounds) to use this test properly. Also remember that the value of the integral is not necessarily the sum of the series.

3. (Direct) Comparison Test:

$$\text{If } \sum b_n \text{ converges and } a_n \leq b_n \text{ for all } n, \text{ then } \sum a_n \text{ converges.}$$
$$\text{If } \sum b_n \text{ diverges and } a_n \geq b_n \text{ for all } n, \text{ then } \sum a_n \text{ diverges.}$$

4. Limit Comparison Test:

$$\text{If } \lim_{n\to\infty} \frac{b_n}{a_n} \text{ exists and is a nonzero number, then both } \sum a_n \text{ and } \sum b_n \text{ either converge or diverge.}$$

To use the Comparison Tests effectively, you need to notice that a given series "looks like" another series which is more familiar and which you know converges or diverges. Then you try to fit the two series into the hypothesis and the conclusion of one of the above Comparison Tests. If you can do it, then you know the given series does the same thing as the one you compared it to. This is why it is important that you know the geometric, harmonic, and p-series inside-out because the comparison will almost always be made with one of these.

The following test can be used for any series, including those with positive and negative terms.

5. Ratio Test:

$$\text{If } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1, \text{ then } \sum a_n \text{ converges.}$$

$$\text{If } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1, \text{ then } \sum a_n \text{ diverges.}$$

$$\text{If } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1, \text{ no conclusion can be made.}$$

Ratio Test is especially good when the index n is in some power or when factorials appear in the terms of the series. Taking the ratio of two consecutive terms will then result in a lot of nice cancellations.

We now turn to series which might have some negative terms. The most common type is the **alternating series**. As the name suggests, the terms alternate in sign. These series therefore can be written in general as

$$\sum_{n=0}^{\infty} (-1)^n a_n \ ,$$

where all the a_n's are positive. Then we have

7. Alternating Series Test:

If $a_n \geq a_{n+1}$ eventually (after finitely many terms) and if $\lim_{n\to\infty} a_n = 0$, then $\sum (-1)^n a_n$ converges.

A good way to deal with series with negative terms is to test for **absolute convergence**. This means disregard all the minus signs and test the new series of positive terms for convergence. Technically, if the original series $\sum a_n$ has negative terms, try looking at $\sum |a_n|$. If $\sum a_n$ is an alternating series, this just means you can throw out $(-1)^n$ from the series. We then have the following useful theorem:

$$\text{If } \sum |a_n| \text{ converges , so does } \sum a_n.$$

In this case, the sum $\sum a_n$ is said to **converge absolutely**. But as usual, there is a warning: If a series does not converge absolutely, that does not mean it does not converge (see harmonic and alternating harmonic). If it does in fact converge (which you usually figure out using Alternating Series Test), then it is said to **converge conditionally**. On the exams, I will always ask you to check an alternating series for absolute and conditional convergence.