

## Help, it's an integral! Now what?

**Step 1:** Take a deep breath and don't panic. If you can't see how to do it right away, that doesn't mean you can't do it! And if your first approach doesn't succeed, that's perfectly normal; just try a different way.

**Step 2:** First, always check if there's a simple algebraic manipulation – such as splitting up a product or quotient – that will make the integral a lot easier.

Example 1: 
$$\int \frac{\sqrt{x} - 1}{x^2} dx$$

But don't forget that you can only split up sums in the numerator of a fraction, not in its denominator!!  
 $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$ !!!

**Step 3:** Second, always check if it's possible to use substitution. Substitution is one of the simplest techniques, but also one of the easiest to overlook. Look for an expression  $u$  such that your integral splits into a product:

$$(\text{simpler function of } u) \cdot u'.$$

Often a good choice for  $u$  is the "inside" function in a composition of functions. In fact, it's worth trying to substitute  $u$  for such an "inside" function even if  $u'$  doesn't appear in the integral.

Example 2: 
$$\int x^3 e^{x^2} dx$$

It's also worth trying a substitution if it looks like this will turn your integral into a rational function (which you can then integrate by other means). This can happen, for example, when you take  $u = \sqrt{x}$  or  $u = e^x$ . (Can you see why?)

Example 3: 
$$\int \frac{1}{e^x - 2} dx.$$

One common mistake is to take  $u$  to be too complicated. It is always better to choose something simple for  $u$ , like  $\sqrt{x}$  or  $\cos(x)$  or a polynomial. Don't try to do massive substitutions, letting  $u$  equal some very complicated expression! If necessary, you can always get there by making multiple simple substitutions, one after the other.

**Step 4:** OK, let's say your first attempt at substitution didn't work. Now see if your function falls into one of the classes for which we have specialized techniques:

- **Is it a rational function?** If so, you're in luck – the procedure for integrating such functions may be complicated, but it's guaranteed to work! The steps are:
  - If the degree of the numerator is greater than or equal to the degree of the denominator, do long division. (The quotient will be a polynomial; don't forget to include it in your final answer!)
  - Factor the denominator if you can. If the leading coefficient isn't 1, factor it out and take it outside the integral (at the bottom of a fraction, of course).
  - If the denominator factors, do the partial fractions procedure. (Don't forget that when you have a quadratic factor in the denominator, the numerator is not just  $A$  but  $Ax + B$ .)
  - Your function is now split into a sum of fractions whose denominators are either linear or unfactorable quadratics. Deal with each fraction separately:
    - \* linear denominator – easy.
    - \* quadratic denominator – complete the square and do substitution to get it into the form...
    - \*  $\frac{ax+b}{x^2+c}$  – split this into  $\frac{x}{x^2+c}$  and  $\frac{1}{x^2+c}$ . Do the former by regular substitution ( $u = x^2 + c$ ), the latter by trig substitution ( $\frac{x^2}{c} = \tan^2 \theta$ ).

Example 4 (about as bad as it gets): 
$$\int \frac{x^4 + x^2 - x + 15}{x^3 + 2x^2 + 5x} dx$$

- **Does your integral only involve trig functions?** Look out for opportunities to use trig identities:
  - If you're faced with an expression involving lots of sines and cosines, try to use the identity  $\sin^2 x + \cos^2 x = 1$  to turn it into a function of the form

$$\cos x \cdot \text{some expression in just } \sin x \quad \text{or} \quad \sin x \cdot \text{some expression in just } \cos x$$

Then use substitution ( $u = \sin x$  or  $u = \cos x$ ).

Example 5: 
$$\int \frac{\sin^3 x}{\cos^8 x} dx$$

- If this doesn't work, try using the fact that both  $\sin^2 x$  and  $\cos^2 x$  can be expressed in terms of  $\cos 2x$ . And remember that  $1/\cos^2 x$  is just  $\sec^2 x$ , which you know how to integrate.
- If you're faced with an expression involving lots of tangents and secants, either simplify it into sines and cosines or use the fact that  $\sec^2 x = 1 + \tan^2 x = \frac{d(\tan x)}{dx}$ .

Example 6: 
$$\int \tan^3 x \sec^5 x dx$$

Example 7: 
$$\int \sec^4 x dx \quad (\text{hard})$$

- Integration by parts sometimes works on products of sines and cosines, though you might have to apply it more than once. Usually there are better ways, but don't rule it out completely.

- **Could trig substitution be appropriate?** Look for expressions of the form  $x^2 + a$ ,  $x^2 - a$ , or  $a - x^2$  in your integral, especially under square root signs. Substitute  $x = \sqrt{a} \tan \theta$ ,  $x = \sqrt{a} \sec \theta$ , and  $x = \sqrt{a} \cos \theta$  respectively, then use trig identities. Don't forget to simplify your final answer so that it doesn't involve expressions like  $\sin^{-1}(\tan x)$ .

**Step 4:** The one thing we haven't tried yet is integration by parts. This is the obvious choice when your integral is a product of two unrelated functions, one of which (call it  $f(x)$ ) has an easy derivative,  $f'(x)$ , and the other (call it  $g(x)$ ) has an easy antiderivative,  $g(x)$ . The best candidates for the role of  $f(x)$  are polynomials (especially  $x$ ) and  $\ln x$ , since they get simpler when you take the derivative.

If you are able to integrate  $f'(x)g(x)$ , you're in business. If not, remember that integration by parts allows for the following, more complicated possibilities:

- $\int f'(x)g(x) dx$  may still be hard, but simpler than the original. Perhaps another application of integration by parts will do the trick.

Example 8: 
$$\int x^2 \cos(x) dx.$$

- $\int f'(x)g(x) dx$  may be expressible in terms of the original integral  $\int f(x)g'(x) dx$ . If so, call this integral  $Z$  and solve for  $Z$ .

Example 9: For any function  $f(x)$ , find 
$$\int f(x)f'(x) dx.$$

- You can take  $g'(x) = 1$  to obtain  $\int f(x) dx = xf(x) - \int xf'(x) dx$ . This is a possibility worth keeping actively in mind, because it is applicable to integrals where you are otherwise unlikely to think of integration by parts. If you are just trying to integrate a function  $f(x)$  and nothing is working, check if integrating  $xf'(x)$  would be simpler. This is especially likely to be the case if your original function somehow involves  $\ln x$ .

Example 10: 
$$\int (\ln x)^2 dx.$$