Math 205: Summary of line and surface integration

1 Line integrals

Line integral of a surface: Let C be a curve in $\mathbb{R}^2$ given parametrically by $x = x(t)$, $y = y(t)$ for $a \le t \le b$. Then the *line integral of a surface* $z = f(x, y)$ *over* C is given by

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

We can think of the parametrization as a vector equation $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}$, in which case this equation can be written as

$$\int_C f(x,y)\,ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt. \tag{1}$$

Special case: If C is given as $y = f(x)$ so that it can be parametrized by x (i.e the parametrization is $x = x$, $y = f(x)$), then the above formula becomes

$$\int_C f(x,y)\,ds = \int_a^b f(x, y(x))\sqrt{(1 + (y'(x))^2}\,dx. \tag{2}$$

Line integral of a vector field: Let $\mathbf{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ be a vector field in $\mathbb{R}^3$ defined along a curve C parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + +z(t)\mathbf{k}$, $a \le t \le b$. Then the *line integral of* $\mathbf{F}$ *over* C is given by

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_a^b (P(x(t),y(t),z(t))x'(t) + Q(x(t),y(t),z(t))y'(t) + R(x(t),y(t),z(t))z'(t))\,dt.$$

Special case: One can also consider the line integral of a vector field in $\mathbb{R}^2$, in which case the R term in the above formula would not be there.

Fundamental Theorem for line integrals: Let C be given parametrically by $\mathbf{r}(t)$, for $a \le t \le b$, either in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let let f be a function of either two or three variable whose gradient ∇f is continuous. Then

$$\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Note: Along with this theorem go various statements about when a vector field is conservative and when the line integral is independent of path, namely

- $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in some domain if and only if $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every C in that domain;
- If $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path, then $\mathbf{F}$ is conservative;
- If $\mathbf{F} = \langle P, Q\rangle$ is continuous on a simply-connected domain and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on that domain, then $\mathbf{F}$ is conservative.

Note 2: When the curve C is a segment of the x-axis from a to b, then Fundamental Theorem for line integrals reduces to Fundamental Theorem of Calculus.

Green's Theorem: Let C be a positively oriented closed curve in $\mathbb{R}^2$ and let D be the region it encloses. Then

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$$

Here $P\,dx + Q\,dy$ can be thought of as the integrand from the formula for the line integral of a vector field above since $\mathbf{F} \cdot d\mathbf{r} = \langle P, Q \rangle \cdot \langle dx, dy \rangle$.

Note: Green's Theorem can be thought of as the counterpart of the Fundamental Theorem of Calculus for double integrals. Namely, FTC says $\int_a^b F'(x)\,dx = F(b) - F(a)$ and, in both FTC and Green's Theorem, there is a derivative of some function(s) in the integrand on the left, and those function(s) appear in the integrand on the right. Also, the right integral in Green's Theorem is computed over the boundary of the domain, but this is also the case in FTC since the two points a and b are the boundary of the interval $[a, b]$.

2 Surface integrals

Surface integral of a hypersurface: Let S be a surface in $\mathbb{R}^3$ given by a vector equation $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ (or parametrically by $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$) for (u, v) in some domain D. Then the *surface integral of* $f(x, y, z)$ *over* S is given by

$$\int_S f(x, y, z)\,dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v|\,dA.$$

Note: This is a generalization of the line integral formula from equation (1).

Special case: If S is given as $z = g(x, y)$ so that it can be parametrized by x and y (i.e the parametrization is $x = x$, $y = y$, $z = g(x, y)$), then the above formula becomes

$$\int_S f(x, y, z)\,dS = \iint_D f(x, y, g(x, y)) \sqrt{1 + \left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g^2}{\partial y} \right)}\,dx.$$

Note that this is of course a generalization of equation (2).

Surface integral of a vector field: Let $\mathbf{F} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a vector field in $\mathbb{R}^3$ defined on an oriented surface S parametrized by $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + +z(u, v)\mathbf{k}$ for (u, v) in some domain D. Also let $\mathbf{n}$ be the unit normal vector to the surface (defined at each point of the surface). Then the *surface integral (or flux) of* $\mathbf{F}$ *over* S is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot |\mathbf{r}_u \times \mathbf{r}_v|\,dA.$$

Special case: If S is given as $z = g(x, y)$, the above formula becomes

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA.$$

Stokes' Theorem: Let S be an oriented surface that is bounded by a closed, positively oriented curve C in $\mathbb{R}^3$. Let $\mathbf{F}$ be a vector field defined on a region containing S. Then

$$\iint_D \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Note: When S is a surface that is contained in the xy-plane, Stokes' Theorem reduces to Green's Theorem. Indeed, the right sides in both statements are identical, and if $\mathbf{F}$ has no z component, then $\operatorname{curl} \mathbf{F}$ reduces to $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ so that the left sides are the same as well.