

Name: _____
 Math 205-03: SI Session #14
 Sections 17.3-17.7
 May 8, 2008

Section 1: Some Background Tools

1. Parametrizing Surfaces in \mathbf{R}^3

To describe a **curve** in \mathbf{R}^3 , we use **one** parameter (often t):

e.g. $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle$ describes a helix
 $\mathbf{r}(t) = \langle 1 + 2t, 4 - t, t \rangle$ describes a line

So far, we've only dealt with integrals over curves, but we can also look at integrals over entire surfaces. In order to do this, we need to be able to describe a **surface** in \mathbf{R}^3 : instead of one parameter, we use **two** (sometimes u and v).

Surface	Parametrization
Sphere of radius a centered at the origin	
Cylinder of radius a on the z -axis	
Plane $ax + by + cz = d$ over $[p, q] \times [r, s]$	
The plane containing two position vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ and a point (x_0, y_0, z_0) on the plane	

Tangent plane to a parametric surface $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ at the point (a,b,c)	
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2. Computing Div & Curl

	Curl (f)	Div (f)
Notation	$\nabla \times \mathbf{F}$	$\nabla \cdot \mathbf{F}$
Type of operation	Cross product (<u>c</u> url \Rightarrow <u>c</u> ross) of $\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$ and \mathbf{F}	Dot product (<u>d</u> iv \Rightarrow <u>d</u> ot) of $\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$ and \mathbf{F}
Outputs a...	Vector	Scalar
Represents	Tendency to rotate	Rate of change at a point

Identity: if \mathbf{F} is a vector field with continuous partial derivatives, then $\text{div}(\text{curl}(\mathbf{F})) = \mathbf{0}$.

Example: Determine $\text{curl}(\mathbf{F})$ and $\text{div}(\mathbf{F})$ where $\mathbf{F}(x,y,z) = \cos xz \mathbf{j} - \sin xy \mathbf{k}$.

Section 2: The Real Stuff

1. Conservative Vector Fields

A vector field in \mathbb{R}^2 is conservative iff...	A vector field in \mathbb{R}^3 is conservative iff...

Example 1. Show that $\mathbf{F} = (2x^3y^4 + x)\mathbf{i} + (2x^4y^3 + y)\mathbf{j}$ is conservative, and find a function f such that $\mathbf{F} = \nabla f$.

Example 2. Show $\int_c (1 - ye^{-x}) dx + e^{-x} dy$ is independent of path.

2. Line Integrals Over Curves

Ultimate Goal: Solving $\int_C \mathbf{F} \cdot d\mathbf{r}$ over a curve C on a vector field \mathbf{F}

Before: plugged in a parameterization $\mathbf{r}(t)$ of our curve C to solve this problem

After: if \mathbf{F} is conservative and we know a function f such that $\mathbf{F} = \nabla f$, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Example. Let \mathbf{F} be the vector field defined by $\mathbf{F}(x,y) = [y^2/(1+x^2)] \mathbf{i} + 2y \tan^{-1}x \mathbf{j}$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is described by $\mathbf{r}(t) = t^2 \mathbf{i} + 2t \mathbf{j}$, $0 \leq t \leq 1$.

3. Line Integrals Over Closed Curves

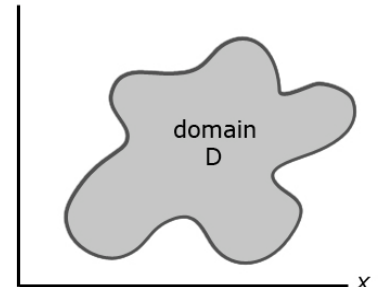
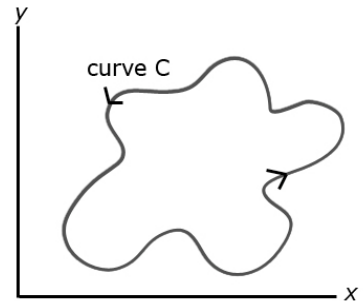
Ultimate Goals:

1. Solving $\int_C P dx + Q dy$ over a closed curve C
2. Solving $\int_C \mathbf{F} \cdot d\mathbf{r}$ over a closed curve C

Before: Parametrized the curve, plugged in this parametrization into the integral, and then solved

After: Can use Green's Theorem: If C is positively oriented, continuous, and has continuous derivatives, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\partial Q/\partial x - \partial P/\partial y) dA = \iint_D \text{curl } \mathbf{F} \cdot \mathbf{k} dA$$



Example 1. Use Green's Theorem to evaluate the line integral $\int_C x e^{-2x} dx + (x^4 + 2x^2 y^2) dy$, where C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Example 2. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle y^2 \cos x, x^2 + 2y \sin x \rangle$, and C is the triangle from (0,0) to (2,6) to (2,0) to (0,0).

4. Integrals & Flux Integrals Over Surfaces

Ultimate Goals:

1. Solving a generic surface integral $\iint_D f(x,y,z) dS$
2. Solving the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ over an oriented surface

$$\text{Formula 1: } \iint_S f(x,y,z) dS = \iint_D f(\mathbf{r}(u,v)) |r_u \times r_v| dA$$

$$\text{Formula 2: if } z = g(x,y), \text{ then}$$
$$\iint_S f(x,y,z) dS = \iint_D f(x,y,g(x,y)) * \text{sqrt}[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1] dA$$

Example 1. Evaluate $\iint_S x^2 z^2 dS$, where S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$.

Warning: Check your orientation! These formulas assume positive orientation.

$$\text{Formula 1: } \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (r_u \times r_v) dA$$

Formula 2: if $z = g(x,y)$, then

$$\iint_S f(x,y,z) dS = \iint_D \mathbf{F}(x, y, g(x,y)) \cdot \langle -\partial z/\partial x, -\partial z/\partial y, 1 \rangle dA$$

Example 2. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + 5\mathbf{k}$ and S is the part of the plane $x + y = 2$ that is enclosed by the cylinder $x^2 + z^2 = 1$, with orientation toward the origin.