

## §31

16. (a) Show that the product space  $\mathbb{R}^I$ , where  $I = [0, 1]$ , has a countable dense subset.
- (b) Show that if  $J$  has cardinality greater than  $\mathcal{P}(\mathbb{Z}_+)$ , then the product space  $\mathbb{R}^J$  does not have a countable dense subset. [Hint: If  $D$  is dense in  $\mathbb{R}^J$ , define  $f : J \rightarrow \mathcal{P}(D)$  by the equation  $f(\alpha) = D \cap \pi_\alpha^{-1}((a, b))$ , where  $(a, b)$  is a fixed interval in  $\mathbb{R}$ .]
- \*17. Give  $\mathbb{R}^\omega$  the box topology. Let  $\mathbb{Q}^\infty$  denote the subspace consisting of sequences of rationals that end in an infinite string of 0's. Which of our four countability axioms does this space satisfy?
- \*18. Let  $G$  be a first-countable topological group. Show that if  $G$  has a countable dense subset, or is Lindelöf, then  $G$  has a countable basis. [Hint: Let  $\{B_n\}$  be a countable basis at  $e$ . If  $D$  is a countable dense subset of  $G$ , show the sets  $dB_n$ , for  $d \in D$ , form a basis for  $G$ . If  $G$  is Lindelöf, choose for each  $n$  a countable set  $C_n$  such that the sets  $cB_n$ , for  $c \in C_n$ , cover  $G$ . Show that as  $n$  ranges over  $\mathbb{Z}_+$ , these sets form a basis for  $G$ .]

### §31 The Separation Axioms

In this section, we introduce three separation axioms and explore some of their properties. One you have already seen—the Hausdorff axiom. The others are similar but stronger. As always when we introduce new concepts, we shall examine the relationship between these axioms and the concepts introduced earlier in the book.

Recall that a space  $X$  is said to be *Hausdorff* if for each pair  $x, y$  of distinct points of  $X$ , there exist disjoint open sets containing  $x$  and  $y$ , respectively.

**Definition.** Suppose that one-point sets are closed in  $X$ . Then  $X$  is said to be *regular* if for each pair consisting of a point  $x$  and a closed set  $B$  disjoint from  $x$ , there exist disjoint open sets containing  $x$  and  $B$ , respectively. The space  $X$  is said to be *normal* if for each pair  $A, B$  of disjoint closed sets of  $X$ , there exist disjoint open sets containing  $A$  and  $B$ , respectively.

It is clear that a regular space is Hausdorff, and that a normal space is regular. (We need to include the condition that one-point sets be closed as part of the definition of regularity and normality in order for this to be the case. A two-point space in the indiscrete topology satisfies the other part of the definitions of regularity and normality, even though it is not Hausdorff.) For examples showing the regularity axiom stronger than the Hausdorff axiom, and normality stronger than regularity, see Examples 1 and 3.

These axioms are called separation axioms for the reason that they involve “separating” certain kinds of sets from one another by disjoint open sets. We have used the word “separation” before, of course, when we studied connected spaces. But in that case, we were trying to find disjoint open sets *whose union was the entire space*.

The present situation is quite different because the open sets need not satisfy this condition.

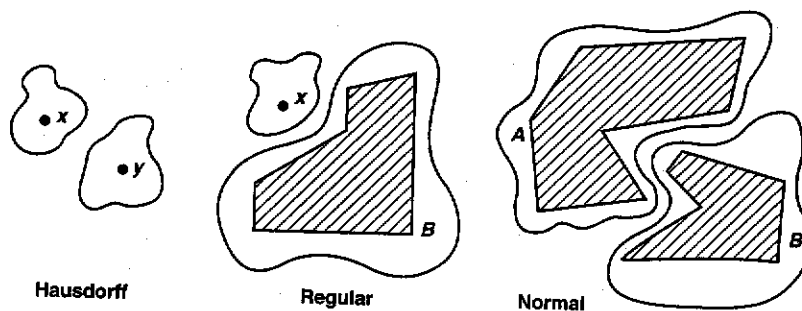


Figure 31.1

The three separation axioms are illustrated in Figure 31.1.

There are other ways to formulate the separation axioms. One formulation that is sometimes useful is given in the following lemma:

**Lemma 31.1.** *Let  $X$  be a topological space. Let one-point sets in  $X$  be closed.*

(a)  *$X$  is regular if and only if given a point  $x$  of  $X$  and a neighborhood  $U$  of  $x$ , there is a neighborhood  $V$  of  $x$  such that  $\bar{V} \subset U$ .*

(b)  *$X$  is normal if and only if given a closed set  $A$  and an open set  $U$  containing  $A$ , there is an open set  $V$  containing  $A$  such that  $\bar{V} \subset U$ .*

*Proof.* (a) Suppose that  $X$  is regular, and suppose that the point  $x$  and the neighborhood  $U$  of  $x$  are given. Let  $B = X - U$ ; then  $B$  is a closed set. By hypothesis, there exist disjoint open sets  $V$  and  $W$  containing  $x$  and  $B$ , respectively. The set  $\bar{V}$  is disjoint from  $B$ , since if  $y \in B$ , the set  $W$  is a neighborhood of  $y$  disjoint from  $V$ . Therefore,  $\bar{V} \subset U$ , as desired.

To prove the converse, suppose the point  $x$  and the closed set  $B$  not containing  $x$  are given. Let  $U = X - B$ . By hypothesis, there is a neighborhood  $V$  of  $x$  such that  $\bar{V} \subset U$ . The open sets  $V$  and  $X - \bar{V}$  are disjoint open sets containing  $x$  and  $B$ , respectively. Thus  $X$  is regular.

(b) This proof uses exactly the same argument; one just replaces the point  $x$  by the set  $A$  throughout. ■

Now we relate the separation axioms with the concepts previously introduced.

**Theorem 31.2.** (a) *A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff.*

(b) *A subspace of a regular space is regular; a product of regular spaces is regular.*

*Proof.* (a) This result was an exercise in §17. We provide a proof here. Let  $X$  be Hausdorff. Let  $x$  and  $y$  be two points of the subspace  $Y$  of  $X$ . If  $U$  and  $V$  are disjoint neighborhoods in  $X$  of  $x$  and  $y$ , respectively, then  $U \cap Y$  and  $V \cap Y$  are disjoint neighborhoods of  $x$  and  $y$  in  $Y$ .

Let  $\{X_\alpha\}$  be a family of Hausdorff spaces. Let  $\mathbf{x} = (x_\alpha)$  and  $\mathbf{y} = (y_\alpha)$  be distinct points of the product space  $\prod X_\alpha$ . Because  $\mathbf{x} \neq \mathbf{y}$ , there is some index  $\beta$  such that  $x_\beta \neq y_\beta$ . Choose disjoint open sets  $U$  and  $V$  in  $X_\beta$  containing  $x_\beta$  and  $y_\beta$ , respectively. Then the sets  $\pi_\beta^{-1}(U)$  and  $\pi_\beta^{-1}(V)$  are disjoint open sets in  $\prod X_\alpha$  containing  $\mathbf{x}$  and  $\mathbf{y}$ , respectively.

(b) Let  $Y$  be a subspace of the regular space  $X$ . Then one-point sets are closed in  $Y$ . Let  $x$  be a point of  $Y$  and let  $B$  be a closed subset of  $Y$  disjoint from  $x$ . Now  $\bar{B} \cap Y = B$ , where  $\bar{B}$  denotes the closure of  $B$  in  $X$ . Therefore,  $x \notin \bar{B}$ , so, using regularity of  $X$ , we can choose disjoint open sets  $U$  and  $V$  of  $X$  containing  $x$  and  $\bar{B}$ , respectively. Then  $U \cap Y$  and  $V \cap Y$  are disjoint open sets in  $Y$  containing  $x$  and  $B$ , respectively.

Let  $\{X_\alpha\}$  be a family of regular spaces; let  $X = \prod X_\alpha$ . By (a),  $X$  is Hausdorff, so that one-point sets are closed in  $X$ . We use the preceding lemma to prove regularity of  $X$ . Let  $\mathbf{x} = (x_\alpha)$  be a point of  $X$  and let  $U$  be a neighborhood of  $\mathbf{x}$  in  $X$ . Choose a basis element  $\prod U_\alpha$  about  $\mathbf{x}$  contained in  $U$ . Choose, for each  $\alpha$ , a neighborhood  $V_\alpha$  of  $x_\alpha$  in  $X_\alpha$  such that  $\bar{V}_\alpha \subset U_\alpha$ ; if it happens that  $U_\alpha = X_\alpha$ , choose  $V_\alpha = X_\alpha$ . Then  $V = \prod V_\alpha$  is a neighborhood of  $\mathbf{x}$  in  $X$ . Since  $\bar{V} = \prod \bar{V}_\alpha$  by Theorem 19.5, it follows at once that  $\bar{V} \subset \prod U_\alpha \subset U$ , so that  $X$  is regular. ■

There is no analogous theorem for normal spaces, as we shall see shortly, in this section and the next.

**EXAMPLE 1.** *The space  $\mathbb{R}_K$  is Hausdorff but not regular.* Recall that  $\mathbb{R}_K$  denotes the reals in the topology having as basis all open intervals  $(a, b)$  and all sets of the form  $(a, b) - K$ , where  $K = \{1/n \mid n \in \mathbb{Z}_+\}$ . This space is Hausdorff, because any two distinct points have disjoint open intervals containing them.

But it is not regular. The set  $K$  is closed in  $\mathbb{R}_K$ , and it does not contain the point 0. Suppose that there exist disjoint open sets  $U$  and  $V$  containing 0 and  $K$ , respectively. Choose a basis element containing 0 and lying in  $U$ . It must be a basis element of the form  $(a, b) - K$ , since each basis element of the form  $(a, b)$  containing 0 intersects  $K$ . Choose  $n$  large enough that  $1/n \in (a, b)$ . Then choose a basis element about  $1/n$  contained in  $V$ ; it must be a basis element of the form  $(c, d)$ . Finally, choose  $z$  so that  $z < 1/n$  and  $z > \max\{c, 1/(n+1)\}$ . Then  $z$  belongs to both  $U$  and  $V$ , so they are not disjoint. See Figure 31.2.

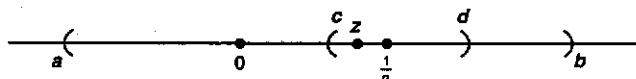


Figure 31.2

EXAMPLE 2. The space  $\mathbb{R}_\ell$  is normal. It is immediate that one-point sets are closed in  $\mathbb{R}_\ell$ , since the topology of  $\mathbb{R}_\ell$  is finer than that of  $\mathbb{R}$ . To check normality, suppose that  $A$  and  $B$  are disjoint closed sets in  $\mathbb{R}_\ell$ . For each point  $a$  of  $A$  choose a basis element  $[a, x_a)$  not intersecting  $B$ ; and for each point  $b$  of  $B$  choose a basis element  $[b, x_b)$  not intersecting  $A$ . The open sets

$$U = \bigcup_{a \in A} [a, x_a) \quad \text{and} \quad V = \bigcup_{b \in B} [b, x_b)$$

are disjoint open sets about  $A$  and  $B$ , respectively.

EXAMPLE 3. The Sorgenfrey plane  $\mathbb{R}_\ell^2$  is not normal.

The space  $\mathbb{R}_\ell$  is regular (in fact, normal), so the product space  $\mathbb{R}_\ell^2$  is also regular. Thus this example serves two purposes. It shows that a regular space need not be normal, and it shows that the product of two normal spaces need not be normal.

We suppose  $\mathbb{R}_\ell^2$  is normal and derive a contradiction. Let  $L$  be the subspace of  $\mathbb{R}_\ell^2$  consisting of all points of the form  $x \times (-x)$ . Then  $L$  is closed in  $\mathbb{R}_\ell^2$ , and  $L$  has the discrete topology. Hence every subset  $A$  of  $L$ , being closed in  $L$ , is closed in  $\mathbb{R}_\ell^2$ . Because  $L - A$  is also closed in  $\mathbb{R}_\ell^2$ , this means that for every nonempty proper subset  $A$  of  $L$ , one can find disjoint open sets  $U_A$  and  $V_A$  containing  $A$  and  $L - A$ , respectively.

Let  $D$  denote the set of points of  $\mathbb{R}_\ell^2$  having rational coordinates; it is dense in  $\mathbb{R}_\ell^2$ . We define a map  $\theta$  that assigns, to each subset of the line  $L$ , a subset of the set  $D$ , by setting

$$\begin{aligned} \theta(A) &= D \cap U_A \quad \text{if } \emptyset \subsetneq A \subsetneq L, \\ \theta(\emptyset) &= \emptyset, \\ \theta(L) &= D. \end{aligned}$$

We show that  $\theta : \mathcal{P}(L) \rightarrow \mathcal{P}(D)$  is injective.

Let  $A$  be a proper nonempty subset of  $L$ . Then  $\theta(A) = D \cap U_A$  is neither empty (since  $U_A$  is open and  $D$  is dense in  $\mathbb{R}_\ell^2$ ) nor all of  $D$  (since  $D \cap V_A$  is nonempty). It remains to show that if  $B$  is another proper nonempty subset of  $L$ , then  $\theta(A) \neq \theta(B)$ .

One of the sets  $A, B$  contains a point not in the other; suppose that  $x \in A$  and  $x \notin B$ . Then  $x \in L - B$ , so that  $x \in U_A \cap V_B$ ; since the latter set is open and nonempty, it must contain points of  $D$ . These points belong to  $U_A$  and not to  $U_B$ ; therefore,  $D \cap U_A \neq D \cap U_B$ , as desired. Thus  $\theta$  is injective.

Now we show there exists an injective map  $\phi : \mathcal{P}(D) \rightarrow L$ . Because  $D$  is countably infinite and  $L$  has the cardinality of  $\mathbb{R}$ , it suffices to define an injective map  $\psi$  of  $\mathcal{P}(\mathbb{Z}_+)$  into  $\mathbb{R}$ . For that, we let  $\psi$  assign to the subset  $S$  of  $\mathbb{Z}_+$  the infinite decimal  $.a_1 a_2 \dots$ , where  $a_i = 0$  if  $i \in S$  and  $a_i = 1$  if  $i \notin S$ . That is,

$$\psi(S) = \sum_{i=1}^{\infty} a_i / 10^i.$$

Now the composite

$$\mathcal{P}(L) \xrightarrow{\theta} \mathcal{P}(D) \xrightarrow{\psi} L$$

is an injective map of  $\mathcal{P}(L)$  into  $L$ . But Theorem 7.8 tells us such a map does not exist! Thus we have reached a contradiction.

This proof that  $\mathbb{R}_l^2$  is not normal is in some ways not very satisfying. We showed only that there must exist some proper nonempty subset  $A$  of  $L$  such that the sets  $A$  and  $B = L - A$  are not contained in disjoint open sets of  $\mathbb{R}_l^2$ . But we did not actually find such a set  $A$ . In fact, the set  $A$  of points of  $L$  having rational coordinates is such a set, but the proof is not easy. It is left to the exercises.

### Exercises

1. Show that if  $X$  is regular, every pair of points of  $X$  have neighborhoods whose closures are disjoint.
2. Show that if  $X$  is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.
3. Show that every order topology is regular.
4. Let  $X$  and  $X'$  denote a single set under two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ , respectively; assume that  $\mathcal{T}' \supset \mathcal{T}$ . If one of the spaces is Hausdorff (or regular, or normal), what does that imply about the other?
5. Let  $f, g : X \rightarrow Y$  be continuous; assume that  $Y$  is Hausdorff. Show that  $\{x \mid f(x) = g(x)\}$  is closed in  $X$ .
6. Let  $p : X \rightarrow Y$  be a closed continuous surjective map. Show that if  $X$  is normal, then so is  $Y$ . [Hint: If  $U$  is an open set containing  $p^{-1}(\{y\})$ , show there is a neighborhood  $W$  of  $y$  such that  $p^{-1}(W) \subset U$ .]
7. Let  $p : X \rightarrow Y$  be a closed continuous surjective map such that  $p^{-1}(\{y\})$  is compact for each  $y \in Y$ . (Such a map is called a *perfect map*.)
  - (a) Show that if  $X$  is Hausdorff, then so is  $Y$ .
  - (b) Show that if  $X$  is regular, then so is  $Y$ .
  - (c) Show that if  $X$  is locally compact, then so is  $Y$ .
  - (d) Show that if  $X$  is second-countable, then so is  $Y$ . [Hint: Let  $\mathcal{B}$  be a countable basis for  $X$ . For each finite subset  $J$  of  $\mathcal{B}$ , let  $U_J$  be the union of all sets of the form  $p^{-1}(W)$ , for  $W$  open in  $Y$ , that are contained in the union of the elements of  $J$ .]
8. Let  $X$  be a space; let  $G$  be a topological group. An *action* of  $G$  on  $X$  is a continuous map  $\alpha : G \times X \rightarrow X$  such that, denoting  $\alpha(g \times x)$  by  $g \cdot x$ , one has:
  - (i)  $e \cdot x = x$  for all  $x \in X$ .
  - (ii)  $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$  for all  $x \in X$  and  $g_1, g_2 \in G$ .
 Define  $x \sim g \cdot x$  for all  $x$  and  $g$ ; the resulting quotient space is denoted  $X/G$  and called the *orbit space* of the action  $\alpha$ .  
**Theorem.** Let  $G$  be a compact topological group; let  $X$  be a topological space; let  $\alpha$  be an action of  $G$  on  $X$ . If  $X$  is Hausdorff, or regular, or normal, or locally compact, or second-countable, so is  $X/G$ .  
 [Hint: See Exercise 13 of §26.]

- \*9. Let  $A$  be the set of all points of  $\mathbb{R}_2^2$  of the form  $x \times (-x)$ , for  $x$  rational; let  $B$  be the set of all points of this form for  $x$  irrational. If  $V$  is an open set of  $\mathbb{R}_2^2$  containing  $B$ , show there exists no open set  $U$  containing  $A$  that is disjoint from  $V$ , as follows:
- Let  $K_n$  consist of all irrational numbers  $x$  in  $[0, 1]$  such that  $[x, x + 1/n) \times [-x, -x + 1/n)$  is contained in  $V$ . Show  $[0, 1]$  is the union of the sets  $K_n$  and countably many one-point sets.
  - Use Exercise 5 of §27 to show that some set  $\bar{K}_n$  contains an open interval  $(a, b)$  of  $\mathbb{R}$ .
  - Show that  $V$  contains the open parallelogram consisting of all points of the form  $x \times (-x + \epsilon)$  for which  $a < x < b$  and  $0 < \epsilon < 1/n$ .
  - Conclude that if  $q$  is a rational number with  $a < q < b$ , then the point  $q \times (-q)$  of  $\mathbb{R}_2^2$  is a limit point of  $V$ .

### §32 Normal Spaces

Now we turn to a more thorough study of spaces satisfying the normality axiom. In one sense, the term "normal" is something of a misnomer, for normal spaces are not as well-behaved as one might wish. On the other hand, most of the spaces with which we are familiar do satisfy this axiom, as we shall see. Its importance comes from the fact that the results one can prove under the hypothesis of normality are central to much of topology. The Urysohn metrization theorem and the Tietze extension theorem are two such results; we shall deal with them later in this chapter.

We begin by proving three theorems that give three important sets of hypotheses under which normality of a space is assured.

**Theorem 32.1.** *Every regular space with a countable basis is normal.*

*Proof.* Let  $X$  be a regular space with a countable basis  $\mathcal{B}$ . Let  $A$  and  $B$  be disjoint closed subsets of  $X$ . Each point  $x$  of  $A$  has a neighborhood  $U$  not intersecting  $B$ . Using regularity, choose a neighborhood  $V$  of  $x$  whose closure lies in  $U$ ; finally, choose an element of  $\mathcal{B}$  containing  $x$  and contained in  $V$ . By choosing such a basis element for each  $x$  in  $A$ , we construct a countable covering of  $A$  by open sets whose closures do not intersect  $B$ . Since this covering of  $A$  is countable, we can index it with the positive integers; let us denote it by  $\{U_n\}$ .

Similarly, choose a countable collection  $\{V_n\}$  of open sets covering  $B$ , such that each set  $\bar{V}_n$  is disjoint from  $A$ . The sets  $U = \bigcup U_n$  and  $V = \bigcup V_n$  are open sets containing  $A$  and  $B$ , respectively, but they need not be disjoint. We perform the following simple trick to construct two open sets that are disjoint. Given  $n$ , define

$$U'_n = U_n - \bigcup_{i=1}^n \bar{V}_i \quad \text{and} \quad V'_n = V_n - \bigcup_{i=1}^n \bar{U}_i.$$

Note that each set  $U'_n$  is open, being the difference of an open set  $U_n$  and a closed set  $\bigcup_{i=1}^n \bar{V}_i$ . Similarly, each set  $V'_n$  is open. The collection  $\{U'_n\}$  covers  $A$ , because each  $x$  in  $A$  belongs to  $U_n$  for some  $n$ , and  $x$  belongs to *none* of the sets  $\bar{V}_i$ . Similarly, the collection  $\{V'_n\}$  covers  $B$ . See Figure 32.1.

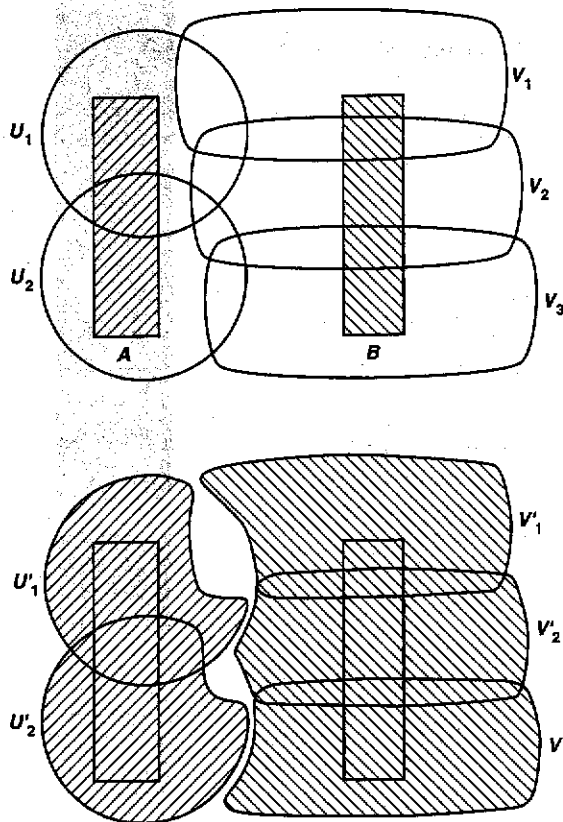


Figure 32.1

Finally, the open sets

$$U' = \bigcup_{n \in \mathbb{Z}_+} U'_n \quad \text{and} \quad V' = \bigcup_{n \in \mathbb{Z}_+} V'_n$$

are disjoint. For if  $x \in U' \cap V'$ , then  $x \in U'_j \cap V'_k$  for some  $j$  and  $k$ . Suppose that  $j \leq k$ . It follows from the definition of  $U'_j$  that  $x \in U_j$ ; and since  $j \leq k$  it follows from the definition of  $V'_k$  that  $x \notin \bar{U}_j$ . A similar contradiction arises if  $j \geq k$ . ■

**Theorem 32.2.** *Every metrizable space is normal.*

*Proof.* Let  $X$  be a metrizable space with metric  $d$ . Let  $A$  and  $B$  be disjoint closed subsets of  $X$ . For each  $a \in A$ , choose  $\epsilon_a$  so that the ball  $B(a, \epsilon_a)$  does not intersect  $B$ . Similarly, for each  $b \in B$ , choose  $\epsilon_b$  so that the ball  $B(b, \epsilon_b)$  does not intersect  $A$ . Define

$$U = \bigcup_{a \in A} B(a, \epsilon_a/2) \quad \text{and} \quad V = \bigcup_{b \in B} B(b, \epsilon_b/2).$$

Then  $U$  and  $V$  are open sets containing  $A$  and  $B$ , respectively; we assert they are disjoint. For if  $z \in U \cap V$ , then

$$z \in B(a, \epsilon_a/2) \cap B(b, \epsilon_b/2)$$

for some  $a \in A$  and some  $b \in B$ . The triangle inequality applies to show that  $d(a, b) < (\epsilon_a + \epsilon_b)/2$ . If  $\epsilon_a \leq \epsilon_b$ , then  $d(a, b) < \epsilon_b$ , so that the ball  $B(b, \epsilon_b)$  contains the point  $a$ . If  $\epsilon_b \leq \epsilon_a$ , then  $d(a, b) < \epsilon_a$ , so that the ball  $B(a, \epsilon_a)$  contains the point  $b$ . Neither situation is possible. ■

**Theorem 32.3.** *Every compact Hausdorff space is normal.*

*Proof.* Let  $X$  be a compact Hausdorff space. We have already essentially proved that  $X$  is regular. For if  $x$  is a point of  $X$  and  $B$  is a closed set in  $X$  not containing  $x$ , then  $B$  is compact, so that Lemma 26.4 applies to show there exist disjoint open sets about  $x$  and  $B$ , respectively.

Essentially the same argument as given in that lemma can be used to show that  $X$  is normal: Given disjoint closed sets  $A$  and  $B$  in  $X$ , choose, for each point  $a$  of  $A$ , disjoint open sets  $U_a$  and  $V_a$  containing  $a$  and  $B$ , respectively. (Here we use regularity of  $X$ .) The collection  $\{U_a\}$  covers  $A$ ; because  $A$  is compact,  $A$  may be covered by finitely many sets  $U_{a_1}, \dots, U_{a_m}$ . Then

$$U = U_{a_1} \cup \dots \cup U_{a_m} \quad \text{and} \quad V = V_{a_1} \cap \dots \cap V_{a_m}$$

are disjoint open sets containing  $A$  and  $B$ , respectively. ■

Here is a further result about normality that we shall find useful in dealing with some examples.

**Theorem 32.4.** *Every well-ordered set  $X$  is normal in the order topology.*

It is, in fact, true that every order topology is normal (see Example 39 of [S-S]); but we shall not have occasion to use this stronger result.

*Proof.* Let  $X$  be a well-ordered set. We assert that every interval of the form  $(x, y]$  is open in  $X$ : If  $X$  has a largest element and  $y$  is that element,  $(x, y]$  is just a basis element about  $y$ . If  $y$  is not the largest element of  $X$ , then  $(x, y]$  equals the open set  $(x, y')$ , where  $y'$  is the immediate successor of  $y$ .

Now let  $A$  and  $B$  be disjoint closed sets in  $X$ ; assume for the moment that neither  $A$  nor  $B$  contains the smallest element  $a_0$  of  $X$ . For each  $a \in A$ , there exists a basis element about  $a$  disjoint from  $B$ ; it contains some interval of the form  $(x, a]$ . (Here is where we use the fact that  $a$  is not the smallest element of  $X$ .) Choose, for each  $a \in A$ , such an interval  $(x_a, a]$  disjoint from  $B$ . Similarly, for each  $b \in B$ , choose an interval  $(y_b, b]$  disjoint from  $A$ . The sets

$$U = \bigcup_{a \in A} (x_a, a] \quad \text{and} \quad V = \bigcup_{b \in B} (y_b, b]$$

are open sets containing  $A$  and  $B$ , respectively; we assert they are disjoint. For suppose that  $z \in U \cap V$ . Then  $z \in (x_a, a] \cap (y_b, b]$  for some  $a \in A$  and some  $b \in B$ . Assume that  $a < b$ . Then if  $a \leq y_b$ , the two intervals are disjoint, while if  $a > y_b$ , we have  $a \in (y_b, b]$ , contrary to the fact that  $(y_b, b]$  is disjoint from  $A$ . A similar contradiction occurs if  $b < a$ .

Finally, assume that  $A$  and  $B$  are disjoint closed sets in  $X$ , and  $A$  contains the smallest element  $a_0$  of  $X$ . The set  $\{a_0\}$  is both open and closed in  $X$ . By the result of the preceding paragraph, there exist disjoint open sets  $U$  and  $V$  containing the closed sets  $A - \{a_0\}$  and  $B$ , respectively. Then  $U \cup \{a_0\}$  and  $V$  are disjoint open sets containing  $A$  and  $B$ , respectively. ■

**EXAMPLE 1.** *If  $J$  is uncountable, the product space  $\mathbb{R}^J$  is not normal.* The proof is fairly difficult; we leave it as a challenging exercise (see Exercise 9).

This example serves three purposes. It shows that a regular space need not be normal. It shows that a subspace of a normal space need not be normal, for  $\mathbb{R}^J$  is homeomorphic to the subspace  $(0, 1)^J$  of  $[0, 1]^J$ , which (assuming the Tychonoff theorem) is compact Hausdorff and therefore normal. And it shows that an uncountable product of normal spaces need not be normal. It leaves unsettled the question as to whether a finite or a countable product of normal spaces might be normal.

**EXAMPLE 2.** *The product space  $S_\Omega \times \bar{S}_\Omega$  is not normal.*<sup>†</sup>

Consider the well-ordered set  $\bar{S}_\Omega$ , in the order topology, and consider the subset  $S_\Omega$ , in the subspace topology (which is the same as the order topology). Both spaces are normal, by Theorem 32.4. We shall show that the product space  $S_\Omega \times \bar{S}_\Omega$  is not normal.

This example serves three purposes. First, it shows that a regular space need not be normal, for  $S_\Omega \times \bar{S}_\Omega$  is a product of regular spaces and therefore regular. Second, it shows that a subspace of a normal space need not be normal, for  $S_\Omega \times \bar{S}_\Omega$  is a subspace of  $\bar{S}_\Omega \times \bar{S}_\Omega$ , which is a compact Hausdorff space and therefore normal. Third, it shows that the product of two normal spaces need not be normal.

First, we consider the space  $\bar{S}_\Omega \times \bar{S}_\Omega$ , and its "diagonal"  $\Delta = \{x \times x \mid x \in \bar{S}_\Omega\}$ . Because  $\bar{S}_\Omega$  is Hausdorff,  $\Delta$  is closed in  $\bar{S}_\Omega \times \bar{S}_\Omega$ : If  $U$  and  $V$  are disjoint neighborhoods of  $x$  and  $y$ , respectively, then  $U \times V$  is a neighborhood of  $x \times y$  that does not intersect  $\Delta$ .

Therefore, in the subspace  $S_\Omega \times \bar{S}_\Omega$ , the set

$$A = \Delta \cap (S_\Omega \times \bar{S}_\Omega) = \Delta - \{\Omega \times \Omega\}$$

<sup>†</sup>Kelley [K] attributes this example to J. Dieudonné and A. P. Morse independently.

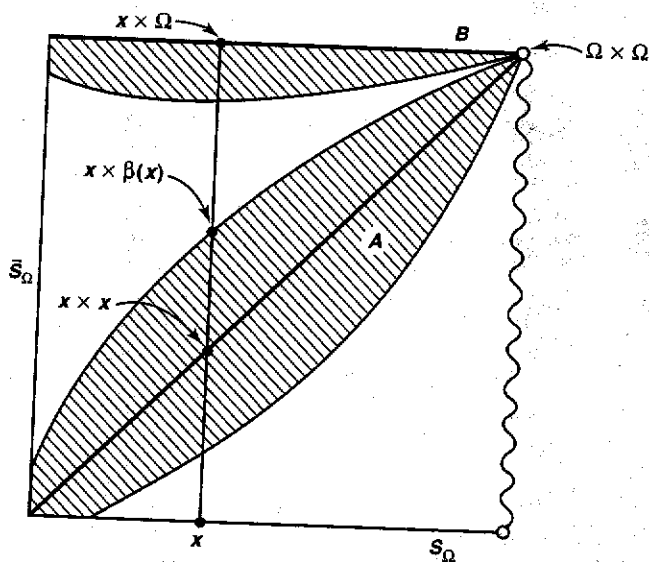


Figure 32.2

is closed. Likewise, the set

$$B = S_\Omega \times \{\Omega\}$$

is closed in  $S_\Omega \times \bar{S}_\Omega$ , being a "slice" of this product space. The sets  $A$  and  $B$  are disjoint. We shall assume there exist disjoint open sets  $U$  and  $V$  of  $S_\Omega \times \bar{S}_\Omega$  containing  $A$  and  $B$ , respectively, and derive a contradiction. See Figure 32.2.

Given  $x \in S_\Omega$ , consider the vertical slice  $x \times \bar{S}_\Omega$ . We assert that there is some point  $\beta$  with  $x < \beta < \Omega$  such that  $x \times \beta$  lies outside  $U$ . For if  $U$  contained all points  $x \times \beta$  for  $x < \beta < \Omega$ , then the top point  $x \times \Omega$  of the slice would be a limit point of  $U$ , which it is not because  $V$  is an open set disjoint from  $U$  containing this top point.

Choose  $\beta(x)$  to be such a point; just to be definite, let  $\beta(x)$  be the *smallest* element of  $S_\Omega$  such that  $x < \beta(x) < \Omega$  and  $x \times \beta(x)$  lies outside  $U$ . Define a sequence of points of  $S_\Omega$  as follows: Let  $x_1$  be any point of  $S_\Omega$ . Let  $x_2 = \beta(x_1)$ , and in general,  $x_{n+1} = \beta(x_n)$ . We have

$$x_1 < x_2 < \dots,$$

because  $\beta(x) > x$  for all  $x$ . The set  $\{x_n\}$  is countable and therefore has an upper bound in  $S_\Omega$ ; let  $b \in S_\Omega$  be its least upper bound. Because the sequence is increasing, it must converge to its least upper bound; thus  $x_n \rightarrow b$ . But  $\beta(x_n) = x_{n+1}$ , so that  $\beta(x_n) \rightarrow b$  also. Then

$$x_n \times \beta(x_n) \rightarrow b \times b$$

in the product space. See Figure 32.3. Now we have a contradiction, for the point  $b \times b$  lies in the set  $A$ , which is contained in the open set  $U$ ; and  $U$  contains *none* of the points  $x_n \times \beta(x_n)$ .

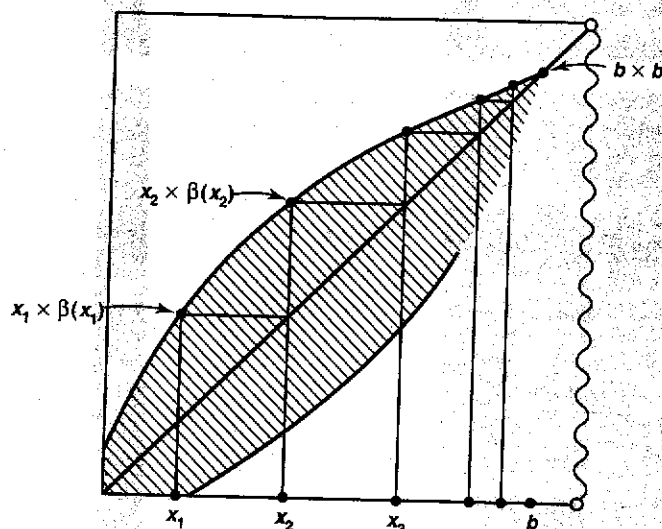


Figure 32.3

### Exercises

1. Show that a closed subspace of a normal space is normal.
2. Show that if  $\prod X_\alpha$  is Hausdorff, or regular, or normal, then so is  $X_\alpha$ . (Assume that each  $X_\alpha$  is nonempty.)
3. Show that every locally compact Hausdorff space is regular.
4. Show that every regular Lindelöf space is normal.
5. Is  $\mathbb{R}^\omega$  normal in the product topology? In the uniform topology?  
It is not known whether  $\mathbb{R}^\omega$  is normal in the box topology. Mary-Ellen Rudin has shown that the answer is affirmative if one assumes the continuum hypothesis [RM]. In fact, she shows it satisfies a stronger condition called *paracompactness*.
6. A space  $X$  is said to be **completely normal** if every subspace of  $X$  is normal. Show that  $X$  is completely normal if and only if for every pair  $A, B$  of separated sets in  $X$  (that is, sets such that  $\bar{A} \cap B = \emptyset$  and  $A \cap \bar{B} = \emptyset$ ), there exist disjoint open sets containing them. [Hint: If  $X$  is completely normal, consider  $X - (\bar{A} \cap B)$ .]
7. Which of the following spaces are completely normal? Justify your answers.
  - (a) A subspace of a completely normal space.
  - (b) The product of two completely normal spaces.
  - (c) A well-ordered set in the order topology.
  - (d) A metrizable space.

- (e) A compact Hausdorff space.
- (f) A regular space with a countable basis.
- (g) The space  $\mathbb{R}_\ell$ .

\*8. Prove the following:

*Theorem.* Every linear continuum  $X$  is normal.

- (a) Let  $C$  be a nonempty closed subset of  $X$ . If  $U$  is a component of  $X - C$ , show that  $U$  is a set of the form  $(c, c')$  or  $(c, \infty)$  or  $(-\infty, c)$ , where  $c, c' \in C$ .
- (b) Let  $A$  and  $B$  be closed disjoint subsets of  $X$ . For each component  $W$  of  $X - A \cup B$  that is an open interval with one end point in  $A$  and the other in  $B$ , choose a point  $c_W$  of  $W$ . Show that the set  $C$  of the points  $c_W$  is closed.
- (c) Show that if  $V$  is a component of  $X - C$ , then  $V$  does not intersect both  $A$  and  $B$ .

\*9. Prove the following:

*Theorem.* If  $J$  is uncountable, then  $\mathbb{R}^J$  is not normal.

*Proof.* (This proof is due to A. H. Stone, as adapted in [S-S].) Let  $X = (\mathbb{Z}_+)^J$ ; it will suffice to show that  $X$  is not normal, since  $X$  is a closed subspace of  $\mathbb{R}^J$ . We use functional notation for the elements of  $X$ , so that the typical element of  $X$  is a function  $x : J \rightarrow \mathbb{Z}_+$ .

- (a) If  $x \in X$  and if  $B$  is a finite subset of  $J$ , let  $U(x, B)$  denote the set consisting of all those elements  $y$  of  $X$  such that  $y(\alpha) = x(\alpha)$  for  $\alpha \in B$ . Show the sets  $U(x, B)$  are a basis for  $X$ .
- (b) Define  $P_n$  to be the subset of  $X$  consisting of those  $x$  such that on the set  $J - x^{-1}(n)$ , the map  $x$  is injective. Show that  $P_1$  and  $P_2$  are closed and disjoint.
- (c) Suppose  $U$  and  $V$  are open sets containing  $P_1$  and  $P_2$ , respectively. Given a sequence  $\alpha_1, \alpha_2, \dots$  of distinct elements of  $J$ , and a sequence

$$0 = n_0 < n_1 < n_2 < \dots$$

of integers, for each  $i \geq 1$  let us set

$$B_i = \{\alpha_1, \dots, \alpha_{n_i}\}$$

and define  $x_i \in X$  by the equations

$$\begin{aligned} x_i(\alpha_j) &= j & \text{for } 1 \leq j \leq n_{i-1}, \\ x_i(\alpha) &= 1 & \text{for all other values of } \alpha. \end{aligned}$$

Show that one can choose the sequences  $\alpha_j$  and  $n_j$  so that for each  $i$ , one has the inclusion

$$U(x_i, B_i) \subset U.$$

[Hint: To begin, note that  $x_1(\alpha) = 1$  for all  $\alpha$ ; now choose  $B_1$  so that  $U(x_1, B_1) \subset U$ .]

- (d) Let  $A$  be the set  $\{\alpha_1, \alpha_2, \dots\}$  constructed in (c). Define  $y: J \rightarrow \mathbb{Z}_+$  by the equations

$$\begin{aligned} y(\alpha_j) &= j && \text{for } \alpha_j \in A, \\ y(\alpha) &= 2 && \text{for all other values of } \alpha. \end{aligned}$$

Choose  $B$  so that  $U(y, B) \subset V$ . Then choose  $i$  so that  $B \cap A$  is contained in the set  $B_i$ . Show that

$$U(x_{i+1}, B_{i+1}) \cap U(y, B)$$

is not empty.

10. Is every topological group normal?

### §33 The Urysohn Lemma

Now we come to the first deep theorem of the book, a theorem that is commonly called the "Urysohn lemma." It asserts the existence of certain real-valued continuous functions on a normal space  $X$ . It is the crucial tool used in proving a number of important theorems. We shall prove three of them—the Urysohn metrization theorem, the Tietze extension theorem, and an imbedding theorem for manifolds—in succeeding sections of this chapter.

Why do we call the Urysohn lemma a "deep" theorem? Because its proof involves a really original idea, which the previous proofs did not. Perhaps we can explain what we mean this way: By and large, one would expect that if one went through this book and deleted all the proofs we have given up to now and then handed the book to a bright student who had not studied topology, that student ought to be able to go through the book and work out the proofs independently. (It would take a good deal of time and effort, of course; and one would not expect the student to handle the trickier examples.) But the Urysohn lemma is on a different level. It would take considerably more originality than most of us possess to prove this lemma unless we were given copious hints!

**Theorem 33.1 (Urysohn lemma).** *Let  $X$  be a normal space; let  $A$  and  $B$  be disjoint closed subsets of  $X$ . Let  $[a, b]$  be a closed interval in the real line. Then there exists a continuous map*

$$f: X \rightarrow [a, b]$$

such that  $f(x) = a$  for every  $x$  in  $A$ , and  $f(x) = b$  for every  $x$  in  $B$ .

*Proof.* We need consider only the case where the interval in question is the interval  $[0, 1]$ ; the general case follows from that one. The first step of the proof is to construct, using normality, a certain family  $U_p$  of open sets of  $X$ , indexed by the rational numbers. Then one uses these sets to define the continuous function  $f$ .

*Step 1.* Let  $P$  be the set of all rational numbers in the interval  $[0, 1]$ .<sup>†</sup> We shall define, for each  $p$  in  $P$ , an open set  $U_p$  of  $X$ , in such a way that whenever  $p < q$ , we have

$$\bar{U}_p \subset U_q.$$

Thus, the sets  $U_p$  will be simply ordered by inclusion in the same way their subscripts are ordered by the usual ordering in the real line.

Because  $P$  is countable, we can use induction to define the sets  $U_p$  (or rather, the principle of recursive definition). Arrange the elements of  $P$  in an infinite sequence in some way; for convenience, let us suppose that the numbers 1 and 0 are the first two elements of the sequence.

Now define the sets  $U_p$ , as follows: First, define  $U_1 = X - B$ . Second, because  $A$  is a closed set contained in the open set  $U_1$ , we may by normality of  $X$  choose an open set  $U_0$  such that

$$A \subset U_0 \quad \text{and} \quad \bar{U}_0 \subset U_1.$$

In general, let  $P_n$  denote the set consisting of the first  $n$  rational numbers in the sequence. Suppose that  $U_p$  is defined for all rational numbers  $p$  belonging to the set  $P_n$ , satisfying the condition

$$(*) \quad p < q \implies \bar{U}_p \subset U_q.$$

Let  $r$  denote the next rational number in the sequence; we wish to define  $U_r$ .

Consider the set  $P_{n+1} = P_n \cup \{r\}$ . It is a finite subset of the interval  $[0, 1]$ , and, as such, it has a simple ordering derived from the usual order relation  $<$  on the real line. In a finite simply ordered set, every element (other than the smallest and the largest) has an immediate predecessor and an immediate successor. (See Theorem 10.1.) The number 0 is the smallest element, and 1 is the largest element, of the simply ordered set  $P_{n+1}$ , and  $r$  is neither 0 nor 1. So  $r$  has an immediate predecessor  $p$  in  $P_{n+1}$  and an immediate successor  $q$  in  $P_{n+1}$ . The sets  $U_p$  and  $U_q$  are already defined, and  $\bar{U}_p \subset U_q$  by the induction hypothesis. Using normality of  $X$ , we can find an open set  $U_r$  of  $X$  such that

$$\bar{U}_p \subset U_r \quad \text{and} \quad \bar{U}_r \subset U_q.$$

We assert that  $(*)$  now holds for every pair of elements of  $P_{n+1}$ . If both elements lie in  $P_n$ ,  $(*)$  holds by the induction hypothesis. If one of them is  $r$  and the other is a point  $s$  of  $P_n$ , then either  $s \leq p$ , in which case

$$\bar{U}_s \subset \bar{U}_p \subset U_r,$$

or  $s \geq q$ , in which case

$$\bar{U}_r \subset U_q \subset U_s.$$

<sup>†</sup>Actually, any countable dense subset of  $[0, 1]$  will do, providing it contains the points 0 and 1.

Thus, for every pair of elements of  $P_{n+1}$ , relation (\*) holds.

By induction, we have  $U_p$  defined for all  $p \in P$ .

To illustrate, let us suppose we started with the standard way of arranging the elements of  $P$  in an infinite sequence:

$$P = \{1, 0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \dots\}$$

After defining  $U_0$  and  $U_1$ , we would define  $U_{1/2}$  so that  $\bar{U}_0 \subset U_{1/2}$  and  $\bar{U}_{1/2} \subset U_1$ . Then we would fit in  $U_{1/3}$  between  $U_0$  and  $U_{1/2}$ ; and  $U_{2/3}$  between  $U_{1/2}$  and  $U_1$ . And so on. At the eighth step of the proof we would have the situation pictured in Figure 33.1. And the ninth step would consist of choosing an open set  $U_{2/5}$  to fit in between  $U_{1/3}$  and  $U_{1/2}$ . And so on.

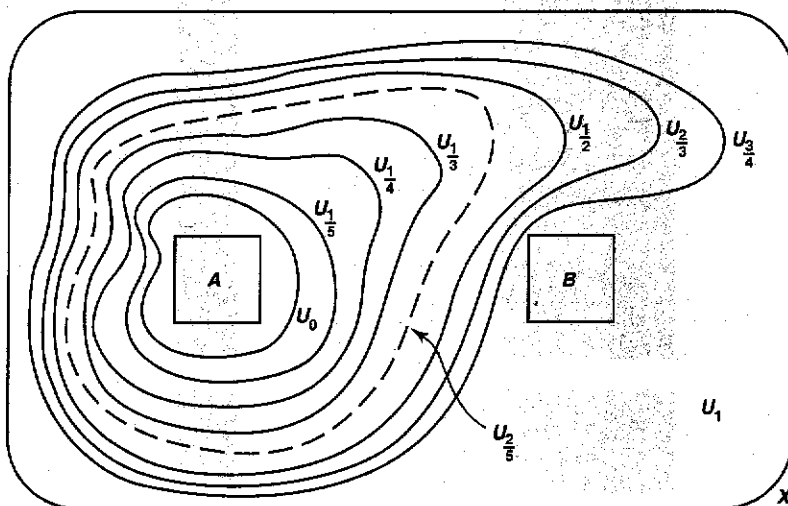


Figure 33.1

Step 2. Now we have defined  $U_p$  for all rational numbers  $p$  in the interval  $[0, 1]$ . We extend this definition to all rational numbers  $p$  in  $\mathbb{R}$  by defining

$$\begin{aligned} U_p &= \emptyset & \text{if } p < 0, \\ U_p &= X & \text{if } p > 1. \end{aligned}$$

It is still true (as you can check) that for any pair of rational numbers  $p$  and  $q$ ,

$$p < q \implies \bar{U}_p \subset U_q.$$

Step 3. Given a point  $x$  of  $X$ , let us define  $\mathbb{Q}(x)$  to be the set of those rational numbers  $p$  such that the corresponding open sets  $U_p$  contain  $x$ :

$$\mathbb{Q}(x) = \{p \mid x \in U_p\}.$$

This set contains no number less than 0, since no  $x$  is in  $U_p$  for  $p < 0$ . And it contains every number greater than 1, since every  $x$  is in  $U_p$  for  $p > 1$ . Therefore,  $\mathbb{Q}(x)$  is bounded below, and its greatest lower bound is a point of the interval  $[0, 1]$ . Define

$$f(x) = \inf \mathbb{Q}(x) = \inf \{p \mid x \in U_p\}.$$

*Step 4.* We show that  $f$  is the desired function. If  $x \in A$ , then  $x \in U_p$  for every  $p \geq 0$ , so that  $\mathbb{Q}(x)$  equals the set of all nonnegative rationals, and  $f(x) = \inf \mathbb{Q}(x) = 0$ . Similarly, if  $x \in B$ , then  $x \in U_p$  for no  $p \leq 1$ , so that  $\mathbb{Q}(x)$  consists of all rational numbers greater than 1, and  $f(x) = 1$ .

All this is easy. The only hard part is to show that  $f$  is continuous. For this purpose, we first prove the following elementary facts:

$$(1) \ x \in \bar{U}_r \Rightarrow f(x) \leq r.$$

$$(2) \ x \notin U_r \Rightarrow f(x) \geq r.$$

To prove (1), note that if  $x \in \bar{U}_r$ , then  $x \in U_s$  for every  $s > r$ . Therefore,  $\mathbb{Q}(x)$  contains all rational numbers greater than  $r$ , so that by definition we have

$$f(x) = \inf \mathbb{Q}(x) \leq r.$$

To prove (2), note that if  $x \notin U_r$ , then  $x$  is not in  $U_s$  for any  $s < r$ . Therefore,  $\mathbb{Q}(x)$  contains no rational numbers less than  $r$ , so that

$$f(x) = \inf \mathbb{Q}(x) \geq r.$$

Now we prove continuity of  $f$ . Given a point  $x_0$  of  $X$  and an open interval  $(c, d)$  in  $\mathbb{R}$  containing the point  $f(x_0)$ , we wish to find a neighborhood  $U$  of  $x_0$  such that  $f(U) \subset (c, d)$ . Choose rational numbers  $p$  and  $q$  such that

$$c < p < f(x_0) < q < d.$$

We assert that the open set

$$U = U_q - \bar{U}_p$$

is the desired neighborhood of  $x_0$ . See Figure 33.2.

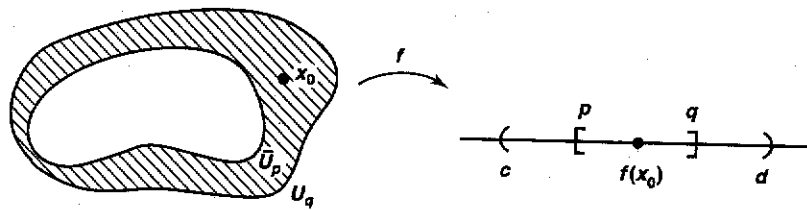


Figure 33.2

First, we note that  $x_0 \in U$ . For the fact that  $f(x_0) < q$  implies by condition (2) that  $x_0 \in U_q$ , while the fact that  $f(x_0) > p$  implies by (1) that  $x_0 \notin \bar{U}_p$ .

Second, we show that  $f(U) \subset (c, d)$ . Let  $x \in U$ . Then  $x \in U_q \subset \bar{U}_q$ , so that  $f(x) \leq q$ , by (1). And  $x \notin \bar{U}_p$ , so that  $x \notin U_p$  and  $f(x) \geq p$ , by (2). Thus,  $f(x) \in [p, q] \subset (c, d)$ , as desired. ■

**Definition.** If  $A$  and  $B$  are two subsets of the topological space  $X$ , and if there is a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(A) = \{0\}$  and  $f(B) = \{1\}$ , we say that  $A$  and  $B$  can be separated by a continuous function.

The Urysohn lemma says that if every pair of disjoint closed sets in  $X$  can be separated by disjoint open sets, then each such pair can be separated by a continuous function. The converse is trivial, for if  $f : X \rightarrow [0, 1]$  is the function, then  $f^{-1}([0, \frac{1}{2}))$  and  $f^{-1}((\frac{1}{2}, 1])$  are disjoint open sets containing  $A$  and  $B$ , respectively.

This fact leads to a question that may already have occurred to you: Why cannot the proof of the Urysohn lemma be generalized to show that in a regular space, where you can separate points from closed sets by disjoint open sets, you can also separate points from closed sets by continuous functions?

At first glance, it seems that the proof of the Urysohn lemma should go through. You take a point  $a$  and a closed set  $B$  not containing  $a$ , and you begin the proof just as before by defining  $U_1 = X - B$  and choosing  $U_0$  to be an open set about  $a$  whose closure is contained in  $U_1$  (using regularity of  $X$ ). But at the very next step of the proof, you run into difficulty. Suppose that  $p$  is the next rational number in the sequence after 0 and 1. You want to find an open set  $U_p$  such that  $\bar{U}_0 \subset U_p$  and  $\bar{U}_p \subset U_1$ . For this, regularity is not enough.

Requiring that one be able to separate a point from a closed set by a continuous function is, in fact, a stronger condition than requiring that one can separate them by disjoint open sets. We make this requirement into a new separation axiom:

**Definition.** A space  $X$  is *completely regular* if one-point sets are closed in  $X$  and if for each point  $x_0$  and each closed set  $A$  not containing  $x_0$ , there is a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(x_0) = 1$  and  $f(A) = \{0\}$ .

A normal space is completely regular, by the Urysohn lemma, and a completely regular space is regular, since given  $f$ , the sets  $f^{-1}([0, \frac{1}{2}))$  and  $f^{-1}((\frac{1}{2}, 1])$  are disjoint open sets about  $A$  and  $x_0$ , respectively. As a result, this new axiom fits in between regularity and normality in the list of separation axioms. Note that in the definition one could just as well require the function to map  $x_0$  to 0, and  $A$  to  $\{1\}$ , for  $g(x) = 1 - f(x)$  satisfies this condition. But our definition is at times a bit more convenient.

In the early years of topology, the separation axioms, listed in order of increasing strength, were labelled  $T_1$ ,  $T_2$  (Hausdorff),  $T_3$  (regular),  $T_4$  (normal), and  $T_5$  (completely normal), respectively. The letter "T" comes from the German "Trennungsaxiom," which means "separation axiom." Later, when the notion of complete regularity was introduced, someone suggested facetiously that it should be called the " $T-3\frac{1}{2}$  axiom," since it lies between regularity and normality. This terminology is in fact sometimes used in the literature!

Unlike normality, this new separation axiom is nicely behaved with regard to subspaces and products:

**Theorem 33.2.** A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular.

*Proof.* Let  $X$  be completely regular; let  $Y$  be a subspace of  $X$ . Let  $x_0$  be a point of  $Y$ , and let  $A$  be a closed set of  $Y$  disjoint from  $x_0$ . Now  $A = \bar{A} \cap Y$ , where  $\bar{A}$  denotes the closure of  $A$  in  $X$ . Therefore,  $x_0 \notin \bar{A}$ . Since  $X$  is completely regular, we can choose a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(x_0) = 1$  and  $f(\bar{A}) = \{0\}$ . The restriction of  $f$  to  $Y$  is the desired continuous function on  $Y$ .

Let  $X = \prod X_\alpha$  be a product of completely regular spaces. Let  $\mathbf{b} = (b_\alpha)$  be a point of  $X$  and let  $A$  be a closed set of  $X$  disjoint from  $\mathbf{b}$ . Choose a basis element  $\prod U_\alpha$  containing  $\mathbf{b}$  that does not intersect  $A$ ; then  $U_\alpha = X_\alpha$  except for finitely many  $\alpha$ , say  $\alpha = \alpha_1, \dots, \alpha_n$ . Given  $i = 1, \dots, n$ , choose a continuous function

$$f_i : X_{\alpha_i} \rightarrow [0, 1]$$

such that  $f_i(b_{\alpha_i}) = 1$  and  $f_i(X - U_{\alpha_i}) = \{0\}$ . Let  $\phi_i(\mathbf{x}) = f_i(\pi_{\alpha_i}(\mathbf{x}))$ ; then  $\phi_i$  maps  $X$  continuously into  $\mathbb{R}$  and vanishes outside  $\pi_{\alpha_i}^{-1}(U_{\alpha_i})$ . The product

$$f(\mathbf{x}) = \phi_1(\mathbf{x}) \cdot \phi_2(\mathbf{x}) \cdot \dots \cdot \phi_n(\mathbf{x})$$

is the desired continuous function on  $X$ , for it equals 1 at  $\mathbf{b}$  and vanishes outside  $\prod U_\alpha$ . ■

**EXAMPLE 1.** The spaces  $\mathbb{R}_\ell^2$  and  $S_\Omega \times \bar{S}_\Omega$  are completely regular but not normal. For they are products of spaces that are completely regular (in fact, normal).

A space that is regular but not completely regular is much harder to find. Most of the examples that have been constructed for this purpose are difficult, and require considerable familiarity with cardinal numbers. Fairly recently, however, John Thomas [T] has constructed a much more elementary example, which we outline in Exercise 11.

## Exercises

1. Examine the proof of the Urysohn lemma, and show that for given  $r$ ,

$$f^{-1}(r) = \bigcap_{p>r} U_p - \bigcup_{q<r} U_q.$$

$p, q$  rational.

2. (a) Show that a connected normal space having more than one point is uncountable.  
(b) Show that a connected regular space having more than one point is uncountable.† [Hint: Any countable space is Lindelöf.]
3. Give a direct proof of the Urysohn lemma for a metric space  $(X, d)$  by setting

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

† Surprisingly enough, there does exist a connected Hausdorff space that is countably infinite. See Example 75 of [S-S].

4. Recall that  $A$  is a " $G_\delta$  set" in  $X$  if  $A$  is the intersection of a countable collection of open sets of  $X$ .

*Theorem.* Let  $X$  be normal. There exists a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(x) = 0$  for  $x \in A$ , and  $f(x) > 0$  for  $x \notin A$ , if and only if  $A$  is a closed  $G_\delta$  set in  $X$ .

A function satisfying the requirements of this theorem is said to *vanish precisely on  $A$* .

5. Prove:

*Theorem (Strong form of the Urysohn lemma).* Let  $X$  be a normal space. There is a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(x) = 0$  for  $x \in A$ , and  $f(x) = 1$  for  $x \in B$ , and  $0 < f(x) < 1$  otherwise, if and only if  $A$  and  $B$  are disjoint closed  $G_\delta$  sets in  $X$ .

6. A space  $X$  is said to be *perfectly normal* if  $X$  is normal and if every closed set in  $X$  is a  $G_\delta$  set in  $X$ .

- (a) Show that every metrizable space is perfectly normal.  
 (b) Show that a perfectly normal space is completely normal. For this reason the condition of perfect normality is sometimes called the " $T_6$  axiom." [Hint: Let  $A$  and  $B$  be separated sets in  $X$ . Choose continuous functions  $f, g : X \rightarrow [0, 1]$  that vanish precisely on  $\bar{A}$  and  $\bar{B}$ , respectively. Consider the function  $f - g$ .]  
 (c) There is a familiar space that is completely normal but not perfectly normal. What is it?

7. Show that every locally compact Hausdorff space is completely regular.

8. Let  $X$  be completely regular; let  $A$  and  $B$  be disjoint closed subsets of  $X$ . Show that if  $A$  is compact, there is a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(A) = \{0\}$  and  $f(B) = \{1\}$ .

9. Show that  $\mathbb{R}^J$  in the box topology is completely regular. [Hint: Show that it suffices to consider the case where the box neighborhood  $(-1, 1)^J$  is disjoint from  $A$  and the point is the origin. Then use the fact that a function continuous in the uniform topology is also continuous in the box topology.]

- \*10. Prove the following:

*Theorem.* Every topological group is completely regular.

*Proof.* Let  $V_0$  be a neighborhood of the identity element  $e$ , in the topological group  $G$ . In general, choose  $V_n$  to be a neighborhood of  $e$  such that  $V_n \cdot V_n \subset V_{n-1}$ . Consider the set of all dyadic rationals  $p$ , that is, all rational numbers of the form  $k/2^n$ , with  $k$  and  $n$  integers. For each dyadic rational  $p$  in  $(0, 1]$ , define an open set  $U(p)$  inductively as follows:  $U(1) = V_0$  and  $U(\frac{1}{2}) = V_1$ . Given  $n$ , if  $U(k/2^n)$  is defined for  $0 < k/2^n \leq 1$ , define

$$U(1/2^{n+1}) = V_{n+1},$$

$$U((2k+1)/2^{n+1}) = V_{n+1} \cdot U(k/2^n)$$

for  $0 < k < 2^n$ . For  $p \leq 0$ , let  $U(p) = \emptyset$ ; and for  $p > 1$ , let  $U(p) = G$ . Show that

$$\bigcup_n U(k/2^n) \subset U((k+1)/2^n)$$

for all  $k$  and  $n$ . Proceed as in the Urysohn lemma.

This exercise is adapted from [M-Z], to which the reader is referred for further results on topological groups.

- \*11. Define a set  $X$  as follows: For each even integer  $m$ , let  $L_m$  denote the line segment  $m \times [-1, 0]$  in the plane. For each odd integer  $n$  and each integer  $k \geq 2$ , let  $C_{n,k}$  denote the union of the line segments  $(n+1-1/k) \times [-1, 0]$  and  $(n-1+1/k) \times [-1, 0]$  and the semicircle

$$\{x \times y \mid (x-n)^2 + y^2 = (1-1/k)^2 \text{ and } y \geq 0\}$$

in the plane. Let  $p_{n,k}$  denote the topmost point  $n \times (1-1/k)$  of this semicircle. Let  $X$  be the union of all the sets  $L_m$  and  $C_{n,k}$ , along with two extra points  $a$  and  $b$ . Topologize  $X$  by taking sets of the following four types as basis elements:

- (i) The intersection of  $X$  with a horizontal open line segment that contains none of the points  $p_{n,k}$ .
  - (ii) A set formed from one of the sets  $C_{n,k}$  by deleting finitely many points.
  - (iii) For each even integer  $m$ , the union of  $\{a\}$  and the set of points  $x \times y$  of  $X$  for which  $x < m$ .
  - (iv) For each even integer  $m$ , the union of  $\{b\}$  and the set of points  $x \times y$  of  $X$  for which  $x > m$ .
- (a) Sketch  $X$ ; show that these sets form a basis for a topology on  $X$ .
  - (b) Let  $f$  be a continuous real-valued function on  $X$ . Show that for any  $c$ , the set  $f^{-1}(c)$  is a  $G_\delta$  set in  $X$ . (This is true for any space  $X$ .) Conclude that the set  $S_{n,k}$  consisting of those points  $p$  of  $C_{n,k}$  for which  $f(p) \neq f(p_{n,k})$  is countable. Choose  $d \in [-1, 0]$  so that the line  $y = d$  intersects none of the sets  $S_{n,k}$ . Show that for  $n$  odd,

$$f((n-1) \times d) = \lim_{k \rightarrow \infty} f(p_{n,k}) = f((n+1) \times d).$$

Conclude that  $f(a) = f(b)$ .

- (c) Show that  $X$  is regular but not completely regular.

### §34 The Urysohn Metrization Theorem

Now we come to the major goal of this chapter, a theorem that gives us conditions under which a topological space is metrizable. The proof weaves together a number of strands from previous parts of the book; it uses results on metric topologies from Chapter 2 as well as facts concerning the countability and separation axioms proved in

the present chapter. The basic construction used in the proof is a simple one, but very useful. You will see it several times more in this book, in various guises.

There are two versions of the proof, and since each has useful generalizations that will appear subsequently, we present both of them here. The first version generalizes to give an imbedding theorem for completely regular spaces. The second version will be generalized in Chapter 6 when we prove the Nagata-Smirnov metrization theorem.

**Theorem 34.1 (Urysohn metrization theorem).** *Every regular space  $X$  with a countable basis is metrizable.*

*Proof.* We shall prove that  $X$  is metrizable by imbedding  $X$  in a metrizable space  $Y$ ; that is, by showing  $X$  homeomorphic with a subspace of  $Y$ . The two versions of the proof differ in the choice of the metrizable space  $Y$ . In the first version,  $Y$  is the space  $\mathbb{R}^\omega$  in the product topology, a space that we have previously proved to be metrizable (Theorem 20.5). In the second version, the space  $Y$  is also  $\mathbb{R}^\omega$ , but this time in the topology given by the uniform metric  $\bar{\rho}$  (see §20). In each case, it turns out that our construction actually imbeds  $X$  in the subspace  $[0, 1]^\omega$  of  $\mathbb{R}^\omega$ .

*Step 1.* We prove the following: *There exists a countable collection of continuous functions  $f_n : X \rightarrow [0, 1]$  having the property that given any point  $x_0$  of  $X$  and any neighborhood  $U$  of  $x_0$ , there exists an index  $n$  such that  $f_n$  is positive at  $x_0$  and vanishes outside  $U$ .*

It is a consequence of the Urysohn lemma that, given  $x_0$  and  $U$ , there exists such a function. However, if we choose one such function for each pair  $(x_0, U)$ , the resulting collection will not in general be countable. Our task is to cut the collection down to size. Here is one way to proceed:

Let  $\{B_n\}$  be a countable basis for  $X$ . For each pair  $n, m$  of indices for which  $\bar{B}_n \subset B_m$ , apply the Urysohn lemma to choose a continuous function  $g_{n,m} : X \rightarrow [0, 1]$  such that  $g_{n,m}(\bar{B}_n) = \{1\}$  and  $g_{n,m}(X - B_m) = \{0\}$ . Then the collection  $\{g_{n,m}\}$  satisfies our requirement: Given  $x_0$  and given a neighborhood  $U$  of  $x_0$ , one can choose a basis element  $B_m$  containing  $x_0$  that is contained in  $U$ . Using regularity, one can then choose  $B_n$  so that  $x_0 \in B_n$  and  $\bar{B}_n \subset B_m$ . Then  $n, m$  is a pair of indices for which the function  $g_{n,m}$  is defined; and it is positive at  $x_0$  and vanishes outside  $U$ . Because the collection  $\{g_{n,m}\}$  is indexed with a subset of  $\mathbb{Z}_+ \times \mathbb{Z}_+$ , it is countable; therefore it can be reindexed with the positive integers, giving us the desired collection  $\{f_n\}$ .

*Step 2 (First version of the proof).* Given the functions  $f_n$  of Step 1, take  $\mathbb{R}^\omega$  in the product topology and define a map  $F : X \rightarrow \mathbb{R}^\omega$  by the rule

$$F(x) = (f_1(x), f_2(x), \dots).$$

We assert that  $F$  is an imbedding.

First,  $F$  is continuous because  $\mathbb{R}^\omega$  has the product topology and each  $f_n$  is continuous. Second,  $F$  is injective because given  $x \neq y$ , we know there is an index  $n$  such that  $f_n(x) > 0$  and  $f_n(y) = 0$ ; therefore,  $F(x) \neq F(y)$ .

Finally, we must prove that  $F$  is a homeomorphism of  $X$  onto its image, the subspace  $Z = F(X)$  of  $\mathbb{R}^\omega$ . We know that  $F$  defines a continuous bijection of  $X$  with  $Z$ ,

so we need only show that for each open set  $U$  in  $X$ , the set  $F(U)$  is open in  $Z$ . Let  $z_0$  be a point of  $F(U)$ . We shall find an open set  $W$  of  $Z$  such that

$$z_0 \in W \subset F(U).$$

Let  $x_0$  be the point of  $U$  such that  $F(x_0) = z_0$ . Choose an index  $N$  for which  $f_N(x_0) > 0$  and  $f_N(X - U) = \{0\}$ . Take the open ray  $(0, +\infty)$  in  $\mathbb{R}$ , and let  $V$  be the open set

$$V = \pi_N^{-1}((0, +\infty))$$

of  $\mathbb{R}^\omega$ . Let  $W = V \cap Z$ ; then  $W$  is open in  $Z$ , by definition of the subspace topology. See Figure 34.1. We assert that  $z_0 \in W \subset F(U)$ . First,  $z_0 \in W$  because

$$\pi_N(z_0) = \pi_N(F(x_0)) = f_N(x_0) > 0.$$

Second,  $W \subset F(U)$ . For if  $z \in W$ , then  $z = F(x)$  for some  $x \in X$ , and  $\pi_N(z) \in (0, +\infty)$ . Since  $\pi_N(z) = \pi_N(F(x)) = f_N(x)$ , and  $f_N$  vanishes outside  $U$ , the point  $x$  must be in  $U$ . Then  $z = F(x)$  is in  $F(U)$ , as desired.

Thus  $F$  is an imbedding of  $X$  in  $\mathbb{R}^\omega$ .

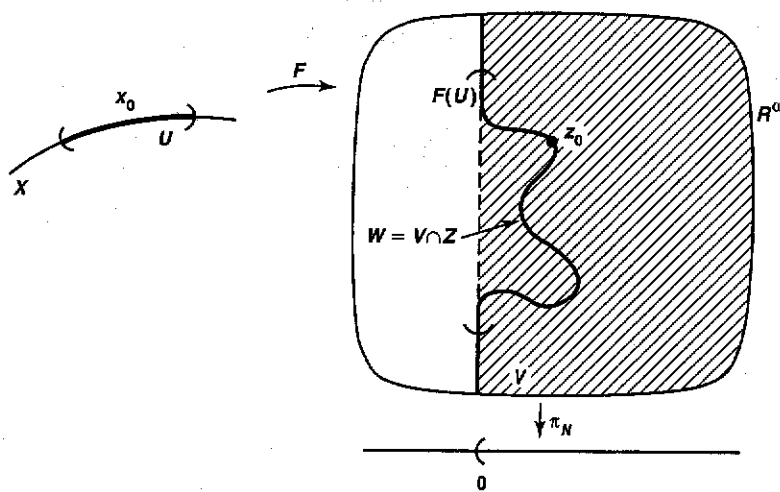


Figure 34.1

*Step 3 (Second version of the proof).* In this version, we imbed  $X$  in the metric space  $(\mathbb{R}^\omega, \bar{\rho})$ . Actually, we imbed  $X$  in the subspace  $[0, 1]^\omega$ , on which  $\bar{\rho}$  equals the metric

$$\rho(\mathbf{x}, \mathbf{y}) = \sup\{|x_i - y_i|\}.$$

We use the countable collection of functions  $f_n : X \rightarrow [0, 1]$  constructed in Step 1. But now we impose the additional condition that  $f_n(x) \leq 1/n$  for all  $x$ . (This condition is easy to satisfy; we can just divide each function  $f_n$  by  $n$ .)

Define  $F : X \rightarrow [0, 1]^\omega$  by the equation

$$F(x) = (f_1(x), f_2(x), \dots)$$

as before. We assert that  $F$  is now an imbedding relative to the metric  $\rho$  on  $[0, 1]^\omega$ . We know from Step 2 that  $F$  is injective. Furthermore, we know that if we use the product topology on  $[0, 1]^\omega$ , the map  $F$  carries open sets of  $X$  onto open sets of the subspace  $Z = F(X)$ . This statement remains true if one passes to the finer (larger) topology on  $[0, 1]^\omega$  induced by the metric  $\rho$ .

The one thing left to do is to prove that  $F$  is continuous. This does not follow from the fact that each component function is continuous, for we are not using the product topology on  $\mathbb{R}^\omega$  now. Here is where the assumption  $f_n(x) \leq 1/n$  comes in.

Let  $x_0$  be a point of  $X$ , and let  $\epsilon > 0$ . To prove continuity, we need to find a neighborhood  $U$  of  $x_0$  such that

$$x \in U \implies \rho(F(x), F(x_0)) < \epsilon.$$

First choose  $N$  large enough that  $1/N \leq \epsilon/2$ . Then for each  $n = 1, \dots, N$  use the continuity of  $f_n$  to choose a neighborhood  $U_n$  of  $x_0$  such that

$$|f_n(x) - f_n(x_0)| \leq \epsilon/2$$

for  $x \in U_n$ . Let  $U = U_1 \cap \dots \cap U_N$ ; we show that  $U$  is the desired neighborhood of  $x_0$ . Let  $x \in U$ . If  $n \leq N$ ,

$$|f_n(x) - f_n(x_0)| \leq \epsilon/2$$

by choice of  $U$ . And if  $n > N$ , then

$$|f_n(x) - f_n(x_0)| < 1/n \leq \epsilon/2$$

because  $f_n$  maps  $X$  into  $[0, 1/n]$ . Therefore for all  $x \in U$ ,

$$\rho(F(x), F(x_0)) \leq \epsilon/2 < \epsilon,$$

as desired. ■

In Step 2 of the preceding proof, we actually proved something stronger than the result stated there. For later use, we state it here:

**Theorem 34.2 (Imbedding theorem).** *Let  $X$  be a space in which one-point sets are closed. Suppose that  $\{f_\alpha\}_{\alpha \in J}$  is an indexed family of continuous functions  $f_\alpha : X \rightarrow \mathbb{R}$  satisfying the requirement that for each point  $x_0$  of  $X$  and each neighborhood  $U$  of  $x_0$ , there is an index  $\alpha$  such that  $f_\alpha$  is positive at  $x_0$  and vanishes outside  $U$ . Then the function  $F : X \rightarrow \mathbb{R}^J$  defined by*

$$F(x) = (f_\alpha(x))_{\alpha \in J}$$

*is an imbedding of  $X$  in  $\mathbb{R}^J$ . If  $f_\alpha$  maps  $X$  into  $[0, 1]$  for each  $\alpha$ , then  $F$  imbeds  $X$  in  $[0, 1]^J$ .*

The proof is almost a copy of Step 2 of the preceding proof; one merely replaces  $n$  by  $\alpha$ , and  $\mathbb{R}^n$  by  $\mathbb{R}^J$ , throughout. One needs one-point sets in  $X$  to be closed in order to be sure that, given  $x \neq y$ , there is an index  $\alpha$  such that  $f_\alpha(x) \neq f_\alpha(y)$ .

A family of continuous functions that satisfies the hypotheses of this theorem is said to *separate points from closed sets* in  $X$ . The existence of such a family is readily seen to be equivalent, for a space  $X$  in which one-point sets are closed, to the requirement that  $X$  be completely regular. Therefore one has the following immediate corollary:

**Theorem 34.3.** *A space  $X$  is completely regular if and only if it is homeomorphic to a subspace of  $[0, 1]^J$  for some  $J$ .*

## Exercises

1. Give an example showing that a Hausdorff space with a countable basis need not be metrizable.
2. Give an example showing that a space can be completely normal, and satisfy the first countability axiom, the Lindelöf condition, and have a countable dense subset, and still not be metrizable.
3. Let  $X$  be a compact Hausdorff space. Show that  $X$  is metrizable if and only if  $X$  has a countable basis.
4. Let  $X$  be a locally compact Hausdorff space. Is it true that if  $X$  has a countable basis, then  $X$  is metrizable? Is it true that if  $X$  is metrizable, then  $X$  has a countable basis?
5. Let  $X$  be a locally compact Hausdorff space. Let  $Y$  be the one-point compactification of  $X$ . Is it true that if  $X$  has a countable basis, then  $Y$  is metrizable? Is it true that if  $Y$  is metrizable, then  $X$  has a countable basis?
6. Check the details of the proof of Theorem 34.2.
7. A space  $X$  is *locally metrizable* if each point  $x$  of  $X$  has a neighborhood that is metrizable in the subspace topology. Show that a compact Hausdorff space  $X$  is metrizable if it is locally metrizable. [Hint: Show that  $X$  is a finite union of open subspaces, each of which has a countable basis.]
8. Show that a regular Lindelöf space is metrizable if it is locally metrizable. [Hint: A closed subspace of a Lindelöf space is Lindelöf.] Regularity is essential; where do you use it in the proof?
9. Let  $X$  be a compact Hausdorff space that is the union of the closed subspaces  $X_1$  and  $X_2$ . If  $X_1$  and  $X_2$  are metrizable, show that  $X$  is metrizable. [Hint: Construct a countable collection  $\mathcal{A}$  of open sets of  $X$  whose intersections with  $X_i$  form a basis for  $X_i$ , for  $i = 1, 2$ . Assume  $X_1 - X_2$  and  $X_2 - X_1$  belong to  $\mathcal{A}$ . Let  $\mathcal{B}$  consist of finite intersections of elements of  $\mathcal{A}$ .]

### \*§35 The Tietze Extension Theorem<sup>†</sup>

One immediate consequence of the Urysohn lemma is the useful theorem called the Tietze extension theorem. It deals with the problem of extending a continuous real-valued function that is defined on a subspace of a space  $X$  to a continuous function defined on all of  $X$ . This theorem is important in many of the applications of topology.

**Theorem 35.1 (Tietze extension theorem).** *Let  $X$  be a normal space; let  $A$  be a closed subspace of  $X$ .*

(a) *Any continuous map of  $A$  into the closed interval  $[a, b]$  of  $\mathbb{R}$  may be extended to a continuous map of all of  $X$  into  $[a, b]$ .*

(b) *Any continuous map of  $A$  into  $\mathbb{R}$  may be extended to a continuous map of all of  $X$  into  $\mathbb{R}$ .*

*Proof.* The idea of the proof is to construct a sequence of continuous functions  $s_n$  defined on the entire space  $X$ , such that the sequence  $s_n$  converges uniformly, and such that the restriction of  $s_n$  to  $A$  approximates  $f$  more and more closely as  $n$  becomes large. Then the limit function will be continuous, and its restriction to  $A$  will equal  $f$ .

*Step 1.* The first step is to construct a particular function  $g$  defined on all of  $X$  such that  $g$  is not too large, and such that  $g$  approximates  $f$  on the set  $A$  to a fair degree of accuracy. To be more precise, let us take the case  $f : A \rightarrow [-r, r]$ . We assert that there exists a continuous function  $g : X \rightarrow \mathbb{R}$  such that

$$\begin{aligned} |g(x)| &\leq \frac{1}{3}r && \text{for all } x \in X, \\ |g(a) - f(a)| &\leq \frac{2}{3}r && \text{for all } a \in A. \end{aligned}$$

The function  $g$  is constructed as follows:

Divide the interval  $[-r, r]$  into three equal intervals of length  $\frac{2}{3}r$ :

$$I_1 = \left[-r, -\frac{1}{3}r\right], \quad I_2 = \left[-\frac{1}{3}r, \frac{1}{3}r\right], \quad I_3 = \left[\frac{1}{3}r, r\right].$$

Let  $B$  and  $C$  be the subsets

$$B = f^{-1}(I_1) \quad \text{and} \quad C = f^{-1}(I_3)$$

of  $A$ . Because  $f$  is continuous,  $B$  and  $C$  are closed disjoint subsets of  $A$ . Therefore, they are closed in  $X$ . By the Urysohn lemma, there exists a continuous function

$$g : X \longrightarrow \left[-\frac{1}{3}r, \frac{1}{3}r\right]$$

having the property that  $g(x) = -\frac{1}{3}r$  for each  $x$  in  $B$ , and  $g(x) = \frac{1}{3}r$  for each  $x$  in  $C$ .

Then  $|g(x)| \leq \frac{1}{3}r$  for all  $x$ . We assert that for each  $a$  in  $A$ ,

$$|g(a) - f(a)| \leq \frac{2}{3}r.$$

<sup>†</sup>This section will be assumed in §62. It is also used in a number of exercises.

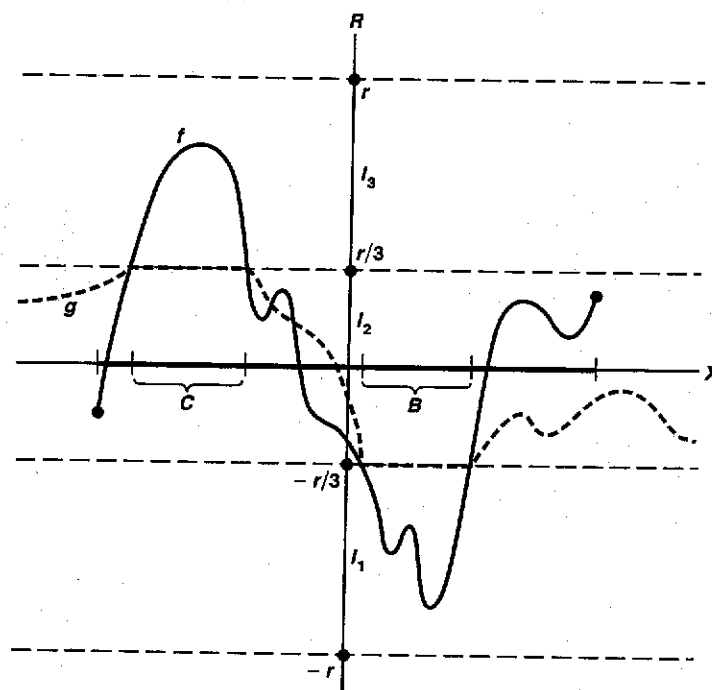


Figure 35.1

There are three cases. If  $a \in B$ , then both  $f(a)$  and  $g(a)$  belong to  $I_1$ . If  $a \in C$ , then  $f(a)$  and  $g(a)$  are in  $I_3$ . And if  $a \notin B \cup C$ , then  $f(a)$  and  $g(a)$  are in  $I_2$ . In each case,  $|g(a) - f(a)| \leq \frac{2}{3}r$ . See Figure 35.1.

*Step 2.* We now prove part (a) of the Tietze theorem. Without loss of generality, we can replace the arbitrary closed interval  $[a, b]$  of  $\mathbb{R}$  by the interval  $[-1, 1]$ .

Let  $f : X \rightarrow [-1, 1]$  be a continuous map. Then  $f$  satisfies the hypotheses of Step 1, with  $r = 1$ . Therefore, there exists a continuous real-valued function  $g_1$ , defined on all of  $X$ , such that

$$\begin{aligned} |g_1(x)| &\leq 1/3 && \text{for } x \in X, \\ |f(a) - g_1(a)| &\leq 2/3 && \text{for } a \in A. \end{aligned}$$

Now consider the function  $f - g_1$ . This function maps  $A$  into the interval  $[-2/3, 2/3]$ , so we can apply Step 1 again, letting  $r = 2/3$ . We obtain a real-valued function  $g_2$

defined on all of  $X$  such that

$$|g_2(x)| \leq \frac{1}{3} \left(\frac{2}{3}\right) \quad \text{for } x \in X,$$

$$|f(a) - g_1(a) - g_2(a)| \leq \left(\frac{2}{3}\right)^2 \quad \text{for } a \in A.$$

Then we apply Step 1 to the function  $f - g_1 - g_2$ . And so on.

At the general step, we have real-valued functions  $g_1, \dots, g_n$  defined on all of  $X$  such that

$$|f(a) - g_1(a) - \dots - g_n(a)| \leq \left(\frac{2}{3}\right)^n$$

for  $a \in A$ . Applying Step 1 to the function  $f - g_1 - \dots - g_n$ , with  $r = \left(\frac{2}{3}\right)^n$ , we obtain a real-valued function  $g_{n+1}$  defined on all of  $X$  such that

$$|g_{n+1}(x)| \leq \frac{1}{3} \left(\frac{2}{3}\right)^n \quad \text{for } x \in X,$$

$$|f(a) - g_1(a) - \dots - g_{n+1}(a)| \leq \left(\frac{2}{3}\right)^{n+1} \quad \text{for } a \in A.$$

By induction, the functions  $g_n$  are defined for all  $n$ .

We now define

$$g(x) = \sum_{n=1}^{\infty} g_n(x)$$

for all  $x$  in  $X$ . Of course, we have to know that this infinite series converges. But that follows from the comparison theorem of calculus; it converges by comparison with the geometric series

$$\frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1}.$$

To show that  $g$  is continuous, we must show that the sequence  $s_n$  converges to  $g$  uniformly. This fact follows at once from the "Weierstrass  $M$ -test" of analysis. Without assuming this result, one can simply note that if  $k > n$ , then

$$|s_k(x) - s_n(x)| = \left| \sum_{i=n+1}^k g_i(x) \right|$$

$$\leq \frac{1}{3} \sum_{i=n+1}^k \left(\frac{2}{3}\right)^{i-1}$$

$$< \frac{1}{3} \sum_{i=n+1}^{\infty} \left(\frac{2}{3}\right)^{i-1} = \left(\frac{2}{3}\right)^n.$$

Holding  $n$  fixed and letting  $k \rightarrow \infty$ , we see that

$$|g(x) - s_n(x)| \leq \left(\frac{2}{3}\right)^n$$

for all  $x \in X$ . Therefore,  $s_n$  converges to  $g$  uniformly.

We show that  $g(a) = f(a)$  for  $a \in A$ . Let  $s_n(x) = \sum_{i=1}^n g_i(x)$ , the  $n$ th partial sum of the series. Then  $g(x)$  is by definition the limit of the infinite sequence  $s_n(x)$  of partial sums. Since

$$|f(a) - \sum_{i=1}^n g_i(a)| = |f(a) - s_n(a)| \leq \left(\frac{2}{3}\right)^n$$

for all  $a$  in  $A$ , it follows that  $s_n(a) \rightarrow f(a)$  for all  $a \in A$ . Therefore, we have  $f(a) = g(a)$  for  $a \in A$ .

Finally, we show that  $g$  maps  $X$  into the interval  $[-1, 1]$ . This condition is in fact satisfied automatically, since the series  $(1/3) \sum (2/3)^n$  converges to 1. However, this is just a lucky accident rather than an essential part of the proof. If all we knew was that  $g$  mapped  $X$  into  $\mathbb{R}$ , then the map  $r \circ g$ , where  $r : \mathbb{R} \rightarrow [-1, 1]$  is the map

$$\begin{aligned} r(y) &= y && \text{if } |y| \leq 1, \\ r(y) &= y/|y| && \text{if } |y| \geq 1, \end{aligned}$$

would be an extension of  $f$  mapping  $X$  into  $[-1, 1]$ .

*Step 3.* We now prove part (b) of the theorem, in which  $f$  maps  $A$  into  $\mathbb{R}$ . We can replace  $\mathbb{R}$  by the open interval  $(-1, 1)$ , since this interval is homeomorphic to  $\mathbb{R}$ .

So let  $f$  be a continuous map from  $A$  into  $(-1, 1)$ . The half of the Tietze theorem already proved shows that we can extend  $f$  to a continuous map  $g : X \rightarrow [-1, 1]$  mapping  $X$  into the *closed* interval. How can we find a map  $h$  carrying  $X$  into the *open* interval?

Given  $g$ , let us define a subset  $D$  of  $X$  by the equation

$$D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\}).$$

Since  $g$  is continuous,  $D$  is a closed subset of  $X$ . Because  $g(A) = f(A)$ , which is contained in  $(-1, 1)$ , the set  $A$  is disjoint from  $D$ . By the Urysohn lemma, there is a continuous function  $\phi : X \rightarrow [0, 1]$  such that  $\phi(D) = \{0\}$  and  $\phi(A) = \{1\}$ . Define

$$h(x) = \phi(x)g(x).$$

Then  $h$  is continuous, being the product of two continuous functions. Also,  $h$  is an extension of  $f$ , since for  $a$  in  $A$ ,

$$h(a) = \phi(a)g(a) = 1 \cdot g(a) = f(a).$$

Finally,  $h$  maps all of  $X$  into the open interval  $(-1, 1)$ . For if  $x \in D$ , then  $h(x) = 0 \cdot g(x) = 0$ . And if  $x \notin D$ , then  $|g(x)| < 1$ ; it follows that  $|h(x)| \leq 1 \cdot |g(x)| < 1$ . ■

## Exercises

1. Show that the Tietze extension theorem implies the Urysohn lemma.
2. In the proof of the Tietze theorem, how essential was the clever decision in Step 1 to divide the interval  $[-r, r]$  into three equal pieces? Suppose instead that one divides this interval into the three intervals

$$I_1 = [-r, -ar], \quad I_2 = [-ar, ar], \quad I_3 = [ar, r],$$

for some  $a$  with  $0 < a < 1$ . For what values of  $a$  other than  $a = 1/3$  (if any) does the proof go through?

3. Let  $X$  be metrizable. Show that the following are equivalent:
  - (i)  $X$  is bounded under every metric that gives the topology of  $X$ .
  - (ii) Every continuous function  $\phi : X \rightarrow \mathbb{R}$  is bounded.
  - (iii)  $X$  is limit point compact.

[Hint: If  $\phi : X \rightarrow \mathbb{R}$  is a continuous function, then  $F(x) = x \times \phi(x)$  is an imbedding of  $X$  in  $X \times \mathbb{R}$ . If  $A$  is an infinite subset of  $X$  having no limit point, let  $\phi$  be a surjection of  $A$  onto  $\mathbb{Z}_+$ .]
4. Let  $Z$  be a topological space. If  $Y$  is a subspace of  $Z$ , we say that  $Y$  is a *retract* of  $Z$  if there is a continuous map  $r : Z \rightarrow Y$  such that  $r(y) = y$  for each  $y \in Y$ .
  - (a) Show that if  $Z$  is Hausdorff and  $Y$  is a retract of  $Z$ , then  $Y$  is closed in  $Z$ .
  - (b) Let  $A$  be a two-point set in  $\mathbb{R}^2$ . Show that  $A$  is not a retract of  $\mathbb{R}^2$ .
  - (c) Let  $S^1$  be the unit circle in  $\mathbb{R}^2$ ; show that  $S^1$  is a retract of  $\mathbb{R}^2 - \{0\}$ , where  $0$  is the origin. Can you conjecture whether or not  $S^1$  is a retract of  $\mathbb{R}^2$ ?
5. A space  $Y$  is said to have the *universal extension property* if for each triple consisting of a normal space  $X$ , a closed subset  $A$  of  $X$ , and a continuous function  $f : A \rightarrow Y$ , there exists an extension of  $f$  to a continuous map of  $X$  into  $Y$ .
  - (a) Show that  $\mathbb{R}^J$  has the universal extension property.
  - (b) Show that if  $Y$  is homeomorphic to a retract of  $\mathbb{R}^J$ , then  $Y$  has the universal extension property.
6. Let  $Y$  be a normal space. Then  $Y$  is said to be an *absolute retract* if for every pair of spaces  $(Y_0, Z)$  such that  $Z$  is normal and  $Y_0$  is a closed subspace of  $Z$  homeomorphic to  $Y$ , the space  $Y_0$  is a retract of  $Z$ .
  - (a) Show that if  $Y$  has the universal extension property, then  $Y$  is an absolute retract.
  - (b) Show that if  $Y$  is an absolute retract and  $Y$  is compact, then  $Y$  has the universal extension property. [Hint: Assume the Tychonoff theorem, so you know  $[0, 1]^J$  is normal. Imbed  $Y$  in  $[0, 1]^J$ .]
7. (a) Show the logarithmic spiral

$$C = \{0 \times 0\} \cup \{e^t \cos t \times e^t \sin t \mid t \in \mathbb{R}\}$$

is a retract of  $\mathbb{R}^2$ . Can you define a specific retraction  $r : \mathbb{R}^2 \rightarrow C$ ?

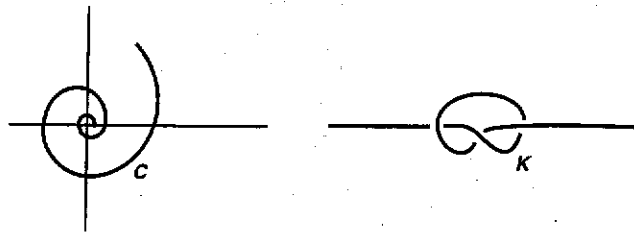


Figure 35.2

(b) Show that the “knotted x-axis”  $K$  of Figure 35.2 is a retract of  $\mathbb{R}^3$ .

\*8. Prove the following:

*Theorem.* Let  $Y$  be a normal space. Then  $Y$  is an absolute retract if and only if  $Y$  has the universal extension property.

[*Hint:* If  $X$  and  $Y$  are disjoint normal spaces,  $A$  is closed in  $X$ , and  $f : A \rightarrow Y$  is a continuous map, define the *adjunction space*  $Z_f$  to be the quotient space obtained from  $X \cup Y$  by identifying each point  $a$  of  $A$  with the point  $f(a)$  and with all the points of  $f^{-1}(\{f(a)\})$ . Using the Tietze theorem, show that  $Z_f$  is normal. If  $p : X \cup Y \rightarrow Z_f$  is the quotient map, show that  $p|_Y$  is a homeomorphism of  $Y$  with a closed subspace of  $Z_f$ .]

9. Let  $X_1 \subset X_2 \subset \dots$  be a sequence of spaces, where  $X_i$  is a closed subspace of  $X_{i+1}$  for each  $i$ . Let  $X$  be the union of the  $X_i$ ; let us topologize  $X$  by declaring a set  $U$  to be open in  $X$  if  $U \cap X_i$  is open in  $X_i$  for each  $i$ .
- Show that this is a topology on  $X$  and that each space  $X_i$  is a subspace (in fact, a closed subspace) of  $X$  in this topology. This topology is called the topology *coherent* with the subspaces  $X_i$ .
  - Show that  $f : X \rightarrow Y$  is continuous if  $f|_{X_i}$  is continuous for each  $i$ .
  - Show that if each space  $X_i$  is normal, then  $X$  is normal. [*Hint:* Given disjoint closed sets  $A$  and  $B$  in  $X$ , set  $f$  equal to 0 on  $A$  and 1 on  $B$ , and extend  $f$  successively to  $A \cup B \cup X_i$  for  $i = 1, 2, \dots$ .]

### \*§36 Imbeddings of Manifolds<sup>†</sup>

We have shown that every regular space with a countable basis can be imbedded in the “infinite-dimensional” euclidean space  $\mathbb{R}^\omega$ . It is natural to ask under what conditions a space  $X$  can be imbedded in some finite-dimensional euclidean space  $\mathbb{R}^N$ . One answer to this question is given in this section. A more general answer will be obtained in Chapter 8, when we study dimension theory.

<sup>†</sup>This section will be assumed when we study paracompactness in §41 and when we study dimension theory in §50.