Math 349 Algebraic Geometry, Spring 2009
Homework 1, due Friday, February 13

(1) Let R be a commutative unital ring. We say that an element $u \in R$ is a *unit* if u has a multiplicative inverse. Prove that the set $U(R) = \{u \in R\colon u \text{ is a unit}\}$ is an abelian group under multiplication.

(2) Using the notation from the previous problem, find the elements of $U(\mathbb{Z}_5)$, $U(\mathbb{Z}_6)$, $U(\mathbb{Z}_{12})$ and $U(\mathbb{Z}_{24})$.

(3) Suppose that S is a set and that R is a ring. Let R^S denote the set of all functions $f\colon S \to R$.
  (a) Prove that R^S is a ring, under the operations defined by the following: for all $f, g \in R^S$ and all $s \in S$, let $(f+g)(s) = f(s)+g(s)$ and $(fg)(s) = f(s)g(s)$. You may assume closure and associativity of both operations.
  (b) Prove that, if S has more than one element, then R^S is not an integral domain.

(4) Prove that an integral domain R with a finite number of elements is a field. (Hint: For each nonzero $a \in R$, consider the map $\lambda_a\colon R \to R$ given by $\lambda_a(r) = ar$ for all $r \in R$. Prove that λ_a is injective and use the fact that any injective function on a finite set is surjective.)

(5) Recall that an ideal I in a commutative unital ring R is *prime* iff $a \in I$ or $b \in I$ whenever $ab \in I$. We say that an element $c \in R$ is *prime* if $c|a$ or $c|b$ whenever $c|ab$. Prove that the following are equivalent for an element $c \in R$:
  (a) the element c is prime in R;
  (b) the ideal $\langle c \rangle$ is prime in R;
  (c) the quotient $R/\langle c \rangle$ is an integral domain.

(6) Suppose that R is an integral domain that is not a field. Prove that $R[x]$ is not a principal ideal domain. (Hint: Let $c \in R$ be nonzero and noninvertible and consider $I = \langle c, x \rangle$.)

(7) Prove that every quotient of a principal ideal domain is also a principal ideal domain. (Hint: Let H be an ideal of R/I and prove that $J = \{a \in R\colon a + I \in H\}$ is an ideal of R and hence principal.)