Math 349 Algebraic Geometry, Spring 2009
Homework 11, due Wednesday, May 13

(1) Let $X = \{x, y\}$ be a two-point topological space with discrete topology. Define a presheaf F as follows: $F(\emptyset) = \emptyset$, $F(\{x\}) = \mathbb{R}$, $F(\{y\}) = \mathbb{R}$, and $F(\{x, y\}) = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The restrictions are defined as

$$\begin{aligned} res_{\{x,y\},\{x\}} \colon F(\{x, y\}) &\longrightarrow F(\{x\}) \\ (x, y, z) &\longmapsto x \\ res_{\{x,y\},\{y\}} \colon F(\{x, y\}) &\longrightarrow F(\{y\}) \\ (x, y, z) &\longmapsto y. \end{aligned}$$

(It is clear what the remaining restriction maps must be.) Show that this is a presheaf but not a sheaf.

(2) Let X be a topological space, x a point in X, and F, G two sheaves on X. Show that a morphism of sheaves $F \to G$ induces a morphism of stalks $F_x \to G_x$.

(3) Show that the field of fractions of a field is isomorphic to the field itself.

(4) Prove that the structure sheaf $\mathcal{O}_V(U)$ of an irreducible affine variety V is actually a sheaf.

(5) (a) Show that $\mathcal{O}_V(V) = k[V]$.
(b) Show that $\mathcal{O}_V(U_1 \cup U_2) = \mathcal{O}_V(U_1) \cap \mathcal{O}_V(U_2)$.

(6) Find $\mathcal{O}_{\mathbb{R}^2}(\mathbb{R}^2 \setminus \{0, 0\})$.

(7) Let R be a commutative ring with unit. Show that the collection of closed sets $\{Z(A) : A \subset R\}$ defined in class gives a topology on $\operatorname{Spec} R$.

(8) Recall that we have bijections

$$\begin{array}{c} \{\text{Zariski closed subsets of } \operatorname{Spec} k[V]\} \\ Z \big\uparrow \big\downarrow Y \\ \{\text{radical ideals of k[V]}\} \\ \mathbb{I} \big\uparrow \big\downarrow \mathbb{V} \\ \{\text{affine subvarieties of } V\} \end{array}$$

Explain why this gives a bijection of open (or closed) sets in $\operatorname{Spec} k[V]$ and in V, i.e. why the topologies on these two spaces are in bijective correspondence.