Calabi-Yau categories, string topology, and Floer field theory

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Report on joint work with Sheel Ganatra

Proof of a conjecture (C., Schwarz, Cielebak - Latchev, Eliashberg) from 2003 relating two topological field theories:

- The string topology of a closed oriented manifold M,
- The Floer - symplectic field theory of its cotangent bundle T^*M.

Background

A symplectic structure on a $2n$-dimensional manifold N is a closed, nondegenerate 2-form, $\omega \in \Omega^2 N$.

For each $x \in N$

$$\omega_x : T_x N \times T_x N \to \mathbb{R}$$

which satisfies

- skew symmetry: $\omega_x(u, v) = -\omega_x(v, u)$
- nondegeracy: $\omega(u, v) = 0$ for all $v \in T_x M$ iff $u = 0$.

Example: $(\mathbb{R}^{2n}, \omega_0)$, where $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$.

Define

$$Sp_{2n} = GL(\mathbb{R}^{2n}, \omega_0)$$

That is, $\psi \in GL_{2n}(\mathbb{R})$ lies in Sp_{2n} iff $\psi^* \omega_0 = \omega_0$.

Now let

$$J_0 = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$$

Then $\psi \in Sp_{2n}$ iff

$$\psi^T J_0 \psi = J_0.$$

Recall that $A \in GL_n(\mathbb{C})$ iff $A^{-1} J_0 A = J_0$. It is now easy to verify the following:

$$Sp_{2n} \cap O(2n) = Sp_{2n} \cap GL_n(\mathbb{C}) = O(2n) \cap GL_n(\mathbb{C}) = U(n).$$

Lemma

$U(n) \subset Sp_{2n}$ is a maximal compact subgroup, and $Sp_{2n}/U(n)$ is contractible.

Corollary

Every symplectic manifold (N, ω) has an almost complex structure, and the space of almost complex structure lifting its given symplectic structure is contractible.

An almost complex structure J compatible with a symplectic structure ω is one in which

$$\begin{aligned} T_xN \times T_xN &\to \mathbb{R} \\ (u, v) &\to \omega_x(u, J_x v) \end{aligned}$$

is a Riemannian metric.

Important Example: Let M^n be closed, $p : T^*M \to M$ its cotangent bundle. For $x \in M$, $u : T_xM \to \mathbb{R}$, define

$$\alpha(x,u) : T_{(x,u)}(T^*M) \xrightarrow{Dp} T_xM \xrightarrow{u} \mathbb{R}$$

$\alpha \in \Omega^1(T^*M)$ is the "Liouville 1-form".

$d\alpha = \omega \in \Omega^2(T^*M)$ is symplectic.

If $N \subset M$ is a submanifold, then its conormal bundle $cn(N) \subset T^*M$ is a Lagrangian submanifold. (A Lagrangian submanifold L of a symplectic manifold Q is defined by the property that $\omega(u,v) = 0$ for all $u, v \in T_xL$.)

Given an exact symplectic manifold (N^{2n}, ω) with $\omega = d\eta$, one can define the Symplectic Floer homology, $SH_*(N, \omega)$. Its defined by doing a type of infinite dimensional Morse theory on the free loop space, LN.

Let $L_0 N \subset LN$ be the path component consisting of null homotopic loops. Consider the symplectic action functional

$$\mathcal{A} : L_0 N \to \mathbb{R}$$
$$\gamma \to \int_{D^2} \tilde{\gamma}^*(\omega)$$

where $\tilde{\gamma} : D^2 \to N$ is an extension (null homotopy) of $\gamma : S^1 \to N$. This is well defined by Stokes' theorem, since

$$\int_{D^2} \tilde{\gamma}^*(\omega) = \int_{S^1} \gamma^*(\eta)$$

(Note if (N, ω) is not exact one can define $\mathcal{A}$ on the universal cover of $L_0 N$.)

One then perturbs $\mathcal{A}$ by a "periodic time dependent Hamiltonian"

$$H : \mathbb{R}/\mathbb{Z} \times N \to \mathbb{R}$$

to get a functional

$$\begin{aligned} \mathcal{A}_H : L_0 N &\to \mathbb{R} \\ \gamma &\to \int_{S^1} \gamma^* \eta - H(t, \gamma(t)) dt \end{aligned} \tag{1}$$

so that $\mathcal{A}_H$ has non degenerate critical points.
If one chooses a compatible almost complex structure J, one has an induced metric, which allows the definition of a Morse-type chain complex (the "Floer complex")

$$\cdots \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \to \cdots$$

The boundary maps

$$\partial[a] = \sum_b n_{a,b}[b]$$

where $n_{a,b} = \#\mathcal{M}(a,b) =$ gradient flow lines (counted with sign).

$$\theta : \mathbb{R} \to LN \quad \text{such that} \quad \frac{d\theta}{dt} + \nabla_J \mathcal{A}_H = 0.$$

(Recall the gradient ∇ depends on the metric, which in this case is given by a choice of J.) If view $\theta : \mathbb{R} \times S^1 \to N$ with coordinates, $t \in \mathbb{R}/\mathbb{Z}$, $s \in \mathbb{R}$, then the gradient flow equation becomes the perturbed Cauchy Riemann PDE:

$$\partial_s \theta - J\partial_t \theta - JX_H(t, \theta(t,s)) = 0.$$

where X_H is the Hamiltonian vector field on $S^1 \times N$ defined by

$$\omega(X_H(t,x), v) = -dH_{(t,x)}(v)$$

"J-pseudoholomorphic cylinders"

Now restrict to the case $(N, \omega) = (T^*M, \omega)$.

Theorem

(Viterbo, Abbondandolo-Schwarz, Salamon-Weber) If M is Spin, then

$$SH_*(T^*M, \omega) \cong H_*(LM).$$

(If M is not spin, one must use twisted coefficients.)

Our goal is to relate two $2D$ open-closed topological field theories.
Both have open boundary conditions defined by closed, oriented submanifolds $\{N \subset M\}$

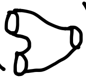
1) String topology of M: $\mathcal{S}_M$

a. $\mathcal{S}_M(S^1) = H_* LM$

b. $\mathcal{S}_M(\underset{N_1}{\bullet}\!\!\longrightarrow\!\!\underset{N_2}{\bullet}) = H_*(\mathcal{P}_M(N_1, N_2))$

where

$$\mathcal{P}_M(N_1,N_2) = \{\gamma : [0,1] \to M : \gamma(0) \in N_1, \ \gamma(1) \in N_2\}$$

c. $\mathcal{S}_M($  $) =$ Chas -Sullivan pairing

$$H_p LM \times H_q LM \longrightarrow H_{p+q-n} LM$$

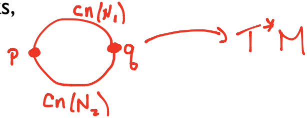
(Chas-Sullivan, C.- Jones, Godin, Kupers)

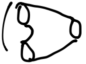
2). Floer symplectic field theory of T*M. Symp_{T^*M}

a. $\text{Symp}_{T^*M}(S^1) = SH_*(T^*M, \omega) \underset{\text{Viterbo}}{\simeq} H_*LM$

b. $\text{Symp}_{T^*M}(\underset{N_1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{N_2}{\bullet}) = HF_*(T^*M; cn(N_1), cn(N_2))$

= "Lagrangian intersection Floer homology"
defined by a chain complex generated by intersection points, $cn(N_1) \cap cn(N_2)$ (if transverse)
boundary homomorphisms defined by counting J-holomorphic disks,



c. $\text{Symp}_{T^*M}($  $)$ $\quad SH_*(T^*M) \times SH_*(T^*M) \longrightarrow SH_*(T^*M)$

Defined by counting J-holomorphic curves

Theorem

(C., Ganatra) Given any field k, there are 2D open-closed, positive boundary, topological field theories, $\mathcal{S}_M$ and $Symp_{T^*M}$ taking values in Chain Complexes over k, such that

1. *When one passes to homology they realize the above theories*
2. *There is a natural equivalence of chain complex valued field theories, $\Phi : Symp_{T^*M} \xrightarrow{\simeq} \mathcal{S}_M$.*

Idea:

Use recent methods of classifying TFT's:

- Cobordism hypothesis of Lurie
- Costello, Kontsevich-Vlassopolous

Roughly: $2D$ "noncompact" ("positive boundary") oriented open-closed TFT's are classified by "Calabi-Yau (A)-∞ categories."

So we show: The string topology category $\mathcal{S}_M$ defined by Blumberg, C., Teleman is Calabi-Yau (actually "Yau-Calabi") as is the "Wrapped Fukaya category" $\mathcal{W}(T^*M)$ defined by Seidel, Fukaya (this part was proved by Ganatra in his thesis) and that

$$\mathcal{S}_M \simeq \mathcal{W}(T^*M)$$

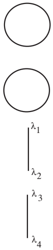
as CY A_∞-categories.

What is a 2D open-closed TFT?

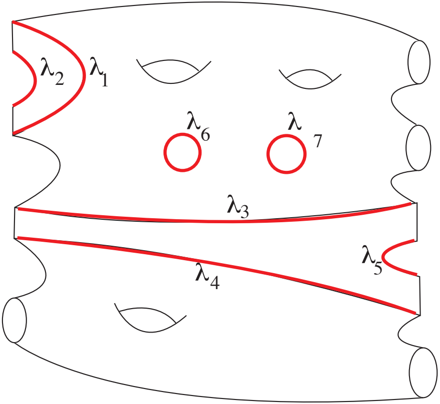
Let $\mathcal{D} = \{N \subset M,\ N \text{ closed, oriented}\}$. Such a field theory is a monoidal functor $\Phi : Bord_{\mathcal{D}}^{oc} \to ChainComplexes$.

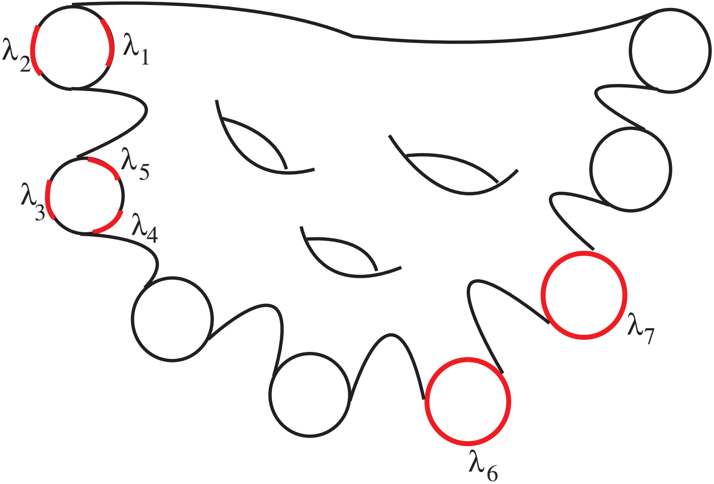
$Bord_{\mathcal{D}}^{oc}$ is a category, enriched over chain complexes:

Objects: Closed, oriented 1-manifolds c, with the path components of ∂c labelled by $\mathcal{D}$.



Morphisms $= C_*(\mathcal{M}^{oc}(c_1, c_2)) =$ chains on the moduli space of oriented open-closed cobordisms:





λ_2
λ_1
λ_5
λ_3
λ_4
λ_7
λ_6

Let A be an (A_∞) algebra over a field k. Consider its Hochschild chains $CH_*(A) \simeq A \otimes^L_{A \otimes A^{op}} A$. It is an ($A_\infty$) module over $E(\Delta) \simeq C_*(S^1)$. The cyclic chains can be viewed as the homotopy orbits $CC_*(A) \simeq CH_*(A) \otimes^L_{E(\Delta)} k$.

Definition

(Kontsevich, Soibelman) Suppose that A is compact (perfect as a k-module). A Calabi-Yau (CY) structure is a map

$$\bar{\tau} : CC_*(A) \to k$$

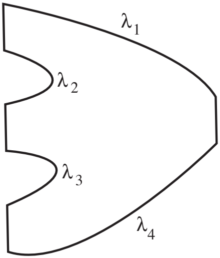
such that the composition
$\tau : A \otimes^L_{A \otimes A^{op}} A \simeq CH_*(A) \to CC_*(A) \xrightarrow{\bar{\tau}} k$ induces a pairing

$$A \otimes A \to k$$

that is homotopy nondegenerate in the sense that the adjoint $A \to A^*$ is an equivalence of A-bimodules. "self duality"

Theorem

(Kontsevich-Soibelman, generalizing Costello) A CY-algebra or category A gives rise to a (left)-positive boundary open-closed field theory $\mathcal{F}_A$ with $\mathcal{F}_A(S^1) \simeq A \otimes^L_{A \otimes A^{op}} A$. The boundary values ("D-branes") of the field theory are $\mathcal{D} = Ob\, A$. The value of $\mathcal{F}$ on the interval with endpoints labeled by $\lambda_1, \lambda_2 \in Ob\, A$ is given by $Mor_A(\lambda_1, \lambda_2)$. The value of $\mathcal{F}_A$ on the open closed cobordism below is given by the higher composition laws in A.



Given an A_∞-algebra or category A, let $CC_*^-(A)$ be the "negative cyclic chains" first defined by Goodwillie. These chains can be viewed as the homotopy fixed points:

$$CC_*^-(A) \simeq Rhom_{E(\Delta)}(k, CH_*(A))$$

- An A_∞ algebra A is said to be "smooth" if is perfect as an A-bimodule. That is, it is perfect as a left module over $A \otimes A^{op}$.
- Let $A^!$ be the "bimodule dual" of A:

$$A^! = Rhom_{A\otimes A^{op}}(A, A \otimes A^{op}) \, (\simeq CH^*(A, A \otimes A^{op}))$$

Definition

(Kontsevich-Vlassopolous) A YC-structure ("Yau-Calabi") on a smooth A_∞-algebra A is an element

$$\bar{\sigma} \in CC_*^-(A)$$

So that if $\sigma \in CH_*(A)$ is the image under the natural map $CC_*^-(A) \to CH_*(A)$, then

$$\cap\sigma : A^! \to A$$

$$Rhom_{A\otimes A^{op}}(A, A \otimes A^{op}) \to A \otimes^L_{A\otimes A^{op}} A \otimes A^{op} \simeq A \qquad (2)$$

is an equivalence of A-bimodules. "self duality as A-bimodules"

Theorem

(Kontsevich-Vlassopolous) A YC-algebra or category A gives rise to a (right)-positive boundary open-closed field theory $\mathcal{F}_A$ with $\mathcal{F}_A(S^1) \simeq A \otimes^L_{A \otimes A^{op}} A$. The boundary values ("D-branes") of the field theory are $\mathcal{D} = Ob\, A$. The value of $\mathcal{F}$ on the interval with endpoints labeled by $\lambda_1, \lambda_2 \in Ob\, A$ is given by $Mor_A(\lambda_1, \lambda_2)$.

Theorem

(C. - Ganatra) The string topology category $\mathcal{S}_M$ and the wrapped Fukaya category $\mathcal{W}(T^*M)$ both have naturally occurring YC-structures whose associated chain complex-valued field theories yield String topology and the Floer-symplectic field theories respectively (on the level of homology). Furthermore there is a natural equivalence $\mathcal{W}(T^*M) \xrightarrow{\simeq} \mathcal{S}_M$ that preserves these YC-structures.

Note: The fact that $\mathcal{W}(T^*M)$ and $\mathcal{S}_M$ are equivalent as A_∞ categories was proved in 2011 by Abouzaid.

Conjecture

(Kontsevich) (maybe proved by Ginzburg) If A is both compact and smooth, then $CY \iff YC$.

Note: In the case where both CY and YC are satisfied, then the field theory is defined on the full cobordism category (i.e no positive boundary condition is required).

Let X be a compact Calabi-Yau variety, then the category of coherent sheaves, $Coh(X)$ is CY. $Coh(X)$ is smooth iff X is smooth. In this case it is also YC. The associated field theory is the "B-model"

Idea of proof Why is there a YC structure on $\mathcal{S}_M$?

Lemma

If $\mathcal{C}_1 \subset \mathcal{C}_2$ generates (i.e the thick subcategory generated by $\mathcal{C}_1$ is $\mathcal{C}_2$), and if both $\mathcal{C}_1$ and $\mathcal{C}_2$ are smooth, then $\mathcal{C}_1$ is YC if and only if $\mathcal{C}_2$ is YC.

Theorem

If M is a closed, oriented n-manifold, the $C_*(\Omega M)$ is YC.

Note: $C_*(\Omega M) = End_{\mathcal{S}_M}(point)$. So by the lemma, this would prove that $\mathcal{S}_M$ is YC.

Sketch of proof. Recall Goodwillie proved that

$$CH_*(C_*(\Omega M) \simeq C_*(LM).$$

Also observe

$$LM^{hS^1} = Map_{S^1}(ES^1, LM) = Map_{S^1}(ES^1 \times S^1, M) \simeq M.$$

So therefore there is a chain map

$$C_*(M) \simeq C_*(LM^{hS^1}) \to Rhom_{C_*(S^1)}(k, CH_*(C_*(\Omega M)) \quad (3)$$
$$= CC_*^-(C_*(\Omega M)). \quad (4)$$

Definition

We say that a cycle $\bar{\sigma} \in CC_*^-(C_*(\Omega M))$ is of *fundamental type* if its homology class $[\bar{\sigma}] \in HC^-(C_*(\Omega M))$ is the image of the fundamental class

$$H_*(M) \to HC_*^-(C_*(\Omega M)) \quad (5)$$
$$[M] \to [\bar{\sigma}]. \quad (6)$$

Claim. Any cycle $\bar{\sigma} \in CC_*^-(C_*(\Omega M))$ of fundamental type defines a YC structure on $C_*(\Omega M)$.

Proof. Let $A = C_*(\Omega M)$. We need to show that if $\sigma \in CH_*(A)$ is the image of $\bar{\sigma} \in CC_*^-(A)$, then

$$\cap\sigma : Rhom_{A\otimes A^{op}}(A, A \otimes A^{op}) \to A$$

is an equivalence.

That is, we need to show

$$\cap[\sigma] : Ext_{A\otimes A^{op}}(A, P) \to Tor_{A\otimes A^{op}}(A, P)$$

is an isomorphism, where $P = A \otimes A^{op}$.
Now since $A = C_*(\Omega M)$ is a connective Hopf algebra, $Ext_{A\otimes A^{op}}(A, P) \cong Ext_A(k, P^{ad})$. (Similarly for Tor).

Since $A = C_*(\Omega M)$ this becomes

$$\cap[\sigma] : H^*(M; P^{ad}) = Ext_{C_*(\Omega M)}(k, P^{ad}) \to Tor_{C_*(\Omega M)}(k, P^{ad}) \tag{7}$$

$$= H_*(M, P^{ad}) \tag{8}$$

(coefficients are twisted by modules over $C_*(\Omega M)$.)

Since $\bar{\sigma}$ is of fundamental type, the fact that this is an isomorphism is Poincaré duality with these twisted coefficients (Dwyer-Greenlees-Iyengar).

Ganatra proved that $\mathcal{W}(T^*M)$ is YC in his thesis. Moreover we have a functor defined by a variant of a construction of Abbondandolo and Schwarz,

$$AS : \mathcal{W}(T^*M) \to \mathcal{S}_M$$

which is seen to be an equivalence of categories by an argument of Abouzaid. Now must check that the YC-structures are preserved. (Technically the most complicated.)

There are two other features.

1. We say that an augmented DGA A is "strongly smooth" if A is smooth and k is a perfect module over A (so in particular $Tor_A(k,k)$ is finite.) $C_*(\Omega M)$ is strongly smooth if M is closed.

Theorem

Let A be a strongly smooth DGA over k. Suppose B is a DGA that is Koszul dual to A. That is,

$$B \simeq Rhom_A(k,k) \quad A \simeq Rhom_B(k,k).$$

They A is YC if and only if B is CY. Furthermore, their associated field theories $\mathcal{F}_A$ and $\mathcal{F}_B$ are dual.

Note: Since A and B are Koszul dual, $HH_*(A) \cong HH_*(B)^*$ (Jones-McCleary) (For THH this is due to J. Campbell.)

Example $A = C_*(\Omega M)$, $B = C^*M$, M simply connected.

Lurie's cobordism hypothesis says that an extended TFT with values in $\mathcal{C}$ (a symmetric monodical $(\infty, 2)$-category) are classified by "Calabi-Yau objects" in $\mathcal{C}$.

Conjecture 1. A is a CY category in the sense of Kontsevich if and only if A is a CY object in the sense of Lurie in the $(\infty, 2)$-category $\mathcal{C}AT$ = Categories, Bimodules, and Maps of Bimodules.

2. A is a YC category in the sense of Kontsevich if and only if A is a CY object in the sense of Lurie in $\mathcal{C}AT^{op}$.

Caution: Need finiteness conditions!

This is a joint project with Ganatra and A. Blumberg.